7.1. Laplace transform.

DEFINITION 7.1 Laplace Transform

Let f be a function defined for $t \geq 0$. Then the integral

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt \qquad (2)$$

is said to be the **Laplace transform*** of f provided the integral converges.

For the most part of Chapter 7, we won't use the definition.
However, there will be problems where we have to use this definition.
So don't forget.

THEOREM 7.2 **Transforms of Some Basic Functions**

(a) $\mathscr{L}\{1\} = \dfrac{1}{s}$

(b) $\mathscr{L}\{t^n\} = \dfrac{n!}{s^{n+1}}, \quad n = 1, 2, 3, \ldots$ (c) $\mathscr{L}\{e^{at}\} = \dfrac{1}{s-a}$

(d) $\mathscr{L}\{\sin kt\} = \dfrac{k}{s^2+k^2}$ (e) $\mathscr{L}\{\cos kt\} = \dfrac{s}{s^2+k^2}$

(f) $\mathscr{L}\{\sinh kt\} = \dfrac{k}{s^2-k^2}$ (g) $\mathscr{L}\{\cosh kt\} = \dfrac{s}{s^2-k^2}$

$\mathscr{L}$ a Linear Transform For a sum of functions we can write

$$\int_0^\infty e^{-st}[\alpha f(t) + \beta g(t)]\, dt = \alpha \int_0^\infty e^{-st} f(t)\, dt + \beta \int_0^\infty e^{-st} g(t)\, dt,$$

whenever both integrals converge. Hence it follows that

$$\mathscr{L}\{\alpha f(t) + \beta g(t)\} = \alpha \mathscr{L}\{f(t)\} + \beta \mathscr{L}\{g(t)\} = \alpha F(s) + \beta G(s). \tag{3}$$

Because of the property given in (3), $\mathscr{L}$ is said to be a **linear transform,** or **linear operator.**

we will use thm 7.2 and linearity property of Laplace transform to evalue Laplace transform of a function

Q: Then when do we have to use the definition?

An: When $f(t)$ is a piecewise function

Example: Evaluate $\mathscr{L}\{f(t)\}$ for $f(t) = \begin{cases} 0 & 0 \le t < 2 \\ 2 & t \ge 2. \end{cases}$

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = \int_0^3 e^{-st}(0)\, dt + \int_3^\infty e^{-st}(2)\, dt$$

$$= \int_0^3 0\, dt + \int_3^\infty 2e^{-st}\, dt = -\frac{2}{s} e^{-st}\Big|_{t=3}^{\infty} = \underbrace{\lim_{t\to\infty} -\frac{2}{s}\frac{1}{e^{st}}}_{e^{st}\to\infty \text{ so, } \frac{1}{e^{st}}\to 0} + \frac{2}{s} e^{-s\cdot 3}$$

$$= 0 + \frac{2}{8} e^{-3s} = 2/se^{3s}, \quad s > 0.$$

($\because$ $s \le 0$, then $\int_0^\infty e^{-st} f(t)\, dt$ will be divergent) e.g.) using linearity property of Laplace transform to find $\mathcal{L}\{f(t)\}$.

$$\mathcal{L}\{\sin^2 t\} = \mathcal{L}\left\{\frac{1-\cos 2t}{2}\right\} = \mathcal{L}\left\{\frac{1}{2}\right\} - \mathcal{L}\left\{\frac{1}{2}\cos 2t\right\}.$$
$$= \frac{1}{2}\mathcal{L}\{1\} - \frac{1}{2}\mathcal{L}\{\cos 2t\}.$$
$$= \frac{1}{2}\cdot\frac{1}{s} - \frac{1}{2}\left(\frac{s}{s^2+2^2}\right) = \frac{1}{2s} - \frac{s}{2s^2+8}$$

e.g). $\mathcal{L}\{e^t \sinh t\} = \mathcal{L}\left\{e^t \frac{e^t - e^{-t}}{2}\right\} = \mathcal{L}\left\{\frac{1}{2}e^t e^t - \frac{1}{2}e^t e^{-t}\right\}$

$* \sinh t = \dfrac{e^t - e^{-t}}{2}$ $\quad = \frac{1}{2}\mathcal{L}\{e^{2t}\} - \frac{1}{2}\mathcal{L}\{e^0\} = \frac{1}{2}\frac{1}{s-2} - \frac{1}{2}\frac{1}{s}$

$* \cosh t = \dfrac{e^t + e^{-t}}{2}$ $\quad = \frac{1}{2s-4} - \frac{1}{2s}.$

e.g). $\mathcal{L}\{(1+2t)^2\} = \mathcal{L}\{1+4t+4t^2\}$

$$= \mathcal{L}\{1\} + \mathcal{L}\{4t\} + \mathcal{L}\{4t^2\}$$
$$= \frac{1}{s} + 4\frac{1!}{s^{1+1}} + 4\frac{2!}{s^{1+2}} = \frac{1}{s} + \frac{4}{s^2} + \frac{8}{s^3}$$

7.2.1 Inverse Transform.

THEOREM 7.3 Some Inverse Transforms

(a) $1 = \mathscr{L}^{-1}\left\{\frac{1}{s}\right\}$

(b) $t^n = \mathscr{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\}, n = 1, 2, 3, \ldots$ (c) $e^{at} = \mathscr{L}^{-1}\left\{\frac{1}{s-a}\right\}$

(d) $\sin kt = \mathscr{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\}$ (e) $\cos kt = \mathscr{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\}$

(f) $\sinh kt = \mathscr{L}^{-1}\left\{\frac{k}{s^2-k^2}\right\}$ (g) $\cosh kt = \mathscr{L}^{-1}\left\{\frac{s}{s^2-k^2}\right\}$

In 7.1, given a function $f(t)$, we evaluate $\mathscr{L}\{f(t)\}$

In 7.2, given $F(s)$, we find $f(t)$ such that $\mathscr{L}\{f(t)\} = F(s)$ or $\mathscr{L}^{-1}\{F(s)\} = f(t)$.

$\mathscr{L}^{-1}$ (inverse Laplace transform) is linear like Laplace transform.

e.g) $\mathscr{L}\{1\} = \frac{1}{s} \longleftrightarrow \mathscr{L}^{-1}\{\frac{1}{s}\} = 1.$

$\mathscr{L}\{\sin 3t\} = \frac{3}{s^2+9} \longleftrightarrow \mathscr{L}^{-1}\{\frac{3}{s^2+9}\} = \sin 3t$

e.g) $\mathscr{L}^{-1}\left\{\frac{4!}{s^{4+1}}\right\} = t^4$ How about $\mathscr{L}^{-1}\left\{\frac{2}{s^{4+1}}\right\} = ?$

linear property

$$\mathscr{L}^{-1}\left\{\frac{2}{s^{4+1}}\right\} = 2\mathscr{L}^{-1}\left\{\frac{1}{s^{4+1}}\right\} = 2\cdot\frac{4!}{4!}\mathscr{L}^{-1}\left\{\frac{1}{s^{4+1}}\right\} = \frac{2}{4!}\mathscr{L}^{-1}\left\{\frac{4!}{s^{4+1}}\right\} = \frac{1}{12}t^4$$

Linearity

e.g.) $\mathcal{L}^{-1}\left\{\frac{1}{s^2+s-20}\right\}$.

$$\frac{1}{s^2+s-20} = \frac{1}{(s+5)(s-4)} = \frac{A}{s+5} + \frac{B}{s-4}$$

$$= \frac{A(s-4)+B(s+5)}{(s+5)(s-4)}$$

$A=-B \qquad -4A+5B=1$

$4B+5B=1 \quad B=\frac{1}{9}.\ A=-\frac{1}{9}$

$$\frac{1}{s^2+s-20} = -\frac{1}{9}\frac{1}{s+5} + \frac{1}{9}\frac{1}{s-4}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+s-20}\right\} = \mathcal{L}^{-1}\left\{-\frac{1}{9}\frac{1}{s+5}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{9}\frac{1}{s-4}\right\}.$$

$$= -\frac{1}{9}\mathcal{L}^{-1}\left\{\frac{1}{s+5}\right\} + \frac{1}{9}\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\}$$

$$= -\frac{1}{9}e^{-5t} + \frac{1}{9}e^{4t}.$$

THEOREM 7.3 **Some Inverse Transforms**

(a) $1 = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}$

(b) $t^n = \mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\}, n = 1, 2, 3, \ldots$

(c) $e^{at} = \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\}$

(d) $\sin kt = \mathcal{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\}$

(e) $\cos kt = \mathcal{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\}$

(f) $\sinh kt = \mathcal{L}^{-1}\left\{\frac{k}{s^2-k^2}\right\}$

(g) $\cosh kt = \mathcal{L}^{-1}\left\{\frac{s}{s^2-k^2}\right\}$

e.g).

$$\mathcal{L}^{-1}\left\{\frac{1}{4s^2+1}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{4(s^2+\frac{1}{4})}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{4}\frac{1}{s^2+(\frac{1}{2})^2}\right\}$$

$$= \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s^2+(\frac{1}{2})^2}\right\}$$

$$= \frac{1}{4}\cdot\frac{\frac{1}{2}}{\frac{1}{2}}\mathcal{L}^{-1}\left\{\frac{1}{s^2+(\frac{1}{2})^2}\right\}$$

$$= \frac{1}{4}\cdot 2\,\mathcal{L}^{-1}\left\{\frac{\frac{1}{2}}{s^2+(\frac{1}{2})^2}\right\}$$

$$= \frac{1}{2}\sin\frac{1}{2}t$$

THEOREM 7.3 **Some Inverse Transforms**

(a) $1 = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}$

(b) $t^n = \mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\}, n = 1, 2, 3, \ldots$ (c) $e^{at} = \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\}$

(d) $\sin kt = \mathcal{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\}$ (e) $\cos kt = \mathcal{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\}$

(f) $\sinh kt = \mathcal{L}^{-1}\left\{\frac{k}{s^2-k^2}\right\}$ (g) $\cosh kt = \mathcal{L}^{-1}\left\{\frac{s}{s^2-k^2}\right\}$

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e.g). $\mathcal{L}^{-1}\left\{\frac{s-1}{s^2+2}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2+2}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{s}{s^2+(\sqrt{2})^2}\right\} - \frac{\sqrt{2}}{\sqrt{2}}\mathcal{L}^{-1}\left\{\frac{1}{s^2+(\sqrt{2})^2}\right\}.$$

$$= \cos\sqrt{2}t - \frac{1}{\sqrt{2}}\mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{s^2+(\sqrt{2})^2}\right\}$$

$$= \cos\sqrt{2}t - \frac{1}{\sqrt{2}}\sin\sqrt{2}t.$$

e.g.) $\mathcal{L}^{-1}\left\{\dfrac{1}{s^4-9}\right\} = \mathcal{L}^{-1}\left\{\dfrac{1}{(s^2+3)(s^2-3)}\right\} = \mathcal{L}^{-1}\left\{\dfrac{As+B}{s^2+3}+\dfrac{Cs+D}{s^2-3}\right\}$

$$\frac{1}{s^4-9} = \frac{As+B}{s^2+3} + \frac{Cs+D}{s^2-3} = \frac{(As+B)(s^2-3)+(Cs+D)(s^2+3)}{(s^2+3)(s^2-3)}$$

$$(As+B)s^2-3(As+B)+(Cs+D)s^2+3(Cs+D)=1$$

$$As^3+Bs^2-3As-3B+Cs^3+Ds^2+3Cs+3D=1$$

$$\left.\begin{aligned} A+C&=0\\ B+D&=0\\ -3A+3C&=0\\ -3B+3D&=1 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} A&=-C\\ A&=C\\ B&=-D\\ D-B&=\tfrac{1}{3} \end{aligned}\right\} \Rightarrow \begin{aligned} A&=0\\ C&=0.\\ \left.\begin{aligned} 2D&=\tfrac{1}{3}\\ B&=-D \end{aligned}\right\} &\Rightarrow \begin{aligned} D&=\tfrac{1}{6}\\ B&=-\tfrac{1}{6} \end{aligned} \end{aligned}$$

$$\frac{1}{s^4-9} = \frac{-\frac{1}{6}}{s^2+3} + \frac{\frac{1}{6}}{s^2-3}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^4-9}\right\} = \mathcal{L}^{-1}\left\{-\frac{1}{6}\,\frac{1}{s^2+3}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{6}\,\frac{1}{s^2-3}\right\}$$

$$= -\frac{1}{6}\,\frac{\sqrt{3}}{\sqrt{3}}\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+(\sqrt{3})^2}\right\} + \frac{1}{6}\,\frac{\sqrt{3}}{\sqrt{3}}\,\mathcal{L}^{-1}\left\{\frac{1}{s^2-(\sqrt{3})^2}\right\}$$

$$= -\frac{1}{6}\,\frac{1}{\sqrt{3}}\,\mathcal{L}^{-1}\left\{\frac{\sqrt{3}}{s^2+(\sqrt{3})^2}\right\} + \frac{1}{6}\,\frac{1}{\sqrt{3}}\,\mathcal{L}^{-1}\left\{\frac{\sqrt{3}}{s^2-(\sqrt{3})^2}\right\}$$

$$= -\frac{\sqrt{3}}{18}\sin\sqrt{3}t + \frac{\sqrt{3}}{18}\sinh kt.$$

e.g.) $\mathcal{L}^{-1}\left\{\frac{s-1}{s^2(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2(s^2+1)}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\}$$

$$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$$

$$1 = A(s^2+1) + (Bs+C)s = As^2 + A + Bs^2 + Cs$$

$$\begin{aligned} A+B&=0 \\ C&=0 \\ A&=1 \end{aligned} \Rightarrow B=-1$$

$$\frac{1}{s(s^2+1)} = \frac{1}{s} + \frac{-s}{s^2+1}$$

$$\frac{1}{s^2(s^2+1)} = \frac{Ds+F}{s^2} + \frac{Gs+E}{s^2+1}$$

$$1 = (Ds+F)(s^2+1) + (Gs+E)s^2$$

$$= Ds^3 + Fs^2 + Ds + F + Gs^3 + Es^2$$

$$\left.\begin{aligned} D+G&=0 \\ F+E&=0 \\ D&=0 \\ F&=1 \end{aligned}\right\} \Rightarrow \begin{aligned} G&=0 \\ E&=-1 \\ D&=0 \\ F&=1 \end{aligned}$$

$$\frac{1}{s^2(s^2+1)} = \frac{1}{s^2} - \frac{1}{s^2+1}$$

$$\mathcal{L}^{-1}\left\{\frac{s-1}{s^2(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= 1 - \mathcal{L}^{-1}\left\{\frac{s}{s^2+1^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1!}{s^{1+1}}\right\} + \sin t$$

$$= 1 - \cos t - t + \sin t.$$

7.2.2 **THEOREM 7.8 Transform of a Derivative**

If $f(t), f'(t), \ldots, f^{(n-1)}(t)$ are continuous on $[0, \infty)$ and are of exponential order and if $f^{n}(t)$ is piecewise continuous on $[0, \infty)$, then

$$\mathscr{L}\{f^{n}(t)\} = s^{n}F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0),$$

where $F(s) = \mathscr{L}\{f(t)\}$.

e.g). $\mathscr{L}\{f'(t)\} = s\mathscr{L}\{f(t)\} - f(0) = sF(s) - f(0).$

$\mathscr{L}\{f''(t)\} = s^2\mathscr{L}\{f(t)\} - sf(0) - f'(0)$

e.g). $\mathscr{L}\{3t\cos 3t + \sin 3t\}$ $\qquad$ * $\frac{d}{dt}(t\sin 3t) = \sin 3t + 3t\cos 3t.$

$= \mathscr{L}\{\frac{d}{dt}t\sin 3t\}$

$= s\mathscr{L}\{t\sin 3t\} - (0\cdot\sin 3\cdot 0)$ $\qquad$ * transform of derivative.

$= s((-1)^1\frac{d}{ds}(\mathscr{L}\{\sin 3t\}))$ $\qquad$ * derivative of transform

$= -s\frac{d}{ds}(\frac{3}{s^2+3^2}) = \frac{s\cdot 3\cdot 2s}{(s^2+3^2)^2}$

7.4.2

Convolution If functions f and g are piecewise continuous on $[0, \infty)$, then the **convolution** of f and g, denoted by $f * g$, is given by the integral

$$f * g = \int_0^t f(\tau)g(t-\tau)\,d\tau.$$

$f*g=g*f$

e.g.) $t * \sin t = \int_0^t \tau \sin(t-\tau)\,d\tau$

$u=\tau \qquad dv=\sin(t-\tau)\,d\tau$

$du=d\tau \qquad v=\cos(t-\tau).$

$= \tau\cos(t-\tau)\Big|_0^t - \int_0^t \cos(t-\tau)\,d\tau$

$= t\cos 0 - 0 - (-\sin(t-\tau))\Big|_0^t$

$= t - \sin(0) - (+\sin t)$

$= t - \sin t.$

THEOREM 7.9 Convolution Theorem

Let $f(t)$ and $g(t)$ be piecewise continuous on $[0, \infty)$ and of exponential order; then

$$\mathscr{L}\{f * g\} = \mathscr{L}\{f(t)\}\mathscr{L}\{g(t)\} = F(s)G(s).$$

* we won't be using this theorem too often.

e.g.) $\mathscr{L}\{\int_0^t \tau \sin(t-\tau)\, d\tau\} = \mathscr{L}\{t * \sin t\}$

$= \mathscr{L}\{t\}\mathscr{L}\{\sin t\} = \frac{1}{s} \cdot \frac{1}{s^2+1}$

Inverse Form of the Convolution Theorem The convolution theorem is sometimes useful in finding the inverse Laplace transform of a product of two Laplace transforms. From Theorem 7.9 we have

$$\mathscr{L}^{-1}\{F(s)G(s)\} = f * g. \tag{6}$$

* we'll use the inverse form of the convolution theorem a lot

* product to convolution

e.g.) $\mathscr{L}^{-1}\{\frac{1}{(s-1)(s+4)}\}$

* Do not use partial fraction decomp.

$= \mathscr{L}^{-1}\{\frac{1}{s-1}\} * \mathscr{L}\{\frac{1}{s+4}\}$

$f(t) = e^{-4t} \quad g(t) = e^{t}$

$= e^{t} * e^{-4t}$

$= \int_0^t e^{-4\tau} e^{t-\tau}\, d\tau$

$= \int_0^t e^{t-5\tau} d\tau = e^t \int_0^t e^{-5\tau} d\tau = e^t \left.-\frac{1}{5} e^{-5\tau}\right|_0^t = e^t \frac{e^{-5t}-1}{-5} = \frac{e^t - e^{-4t}}{5}$

#4 $y(0)=1$. $y'(0)=-1$

what is Laplace transform of $y''-4y'+5y$?

$\mathcal{L}\{y''-4y'+5y\} = \mathcal{L}\{y''\} - 4\mathcal{L}\{y'\} + 5\mathcal{L}\{y\}$

$= s^2\mathcal{L}\{y\} - sy(0) - y'(0) - 4(s\mathcal{L}\{y\} - y(0)) + 5\mathcal{L}\{y\}$.

$= s^2\mathcal{L}\{y\} - s\cdot 1 - (-1) - 4s\mathcal{L}\{y\} + 4\cdot 1 + 5\mathcal{L}\{y\}$

$= \mathcal{L}\{y\}(s^2-4s+5) - s + 5$

$= Y(s)(s^2-4s+5) - s+5$

#6. $y(0)=2$, $y'(0)=3$. Solve for $\mathcal{L}\{y(t)\}=Y(s)$, $y''+y=1$

$$y''+y=1 \longrightarrow \mathcal{L}\{y''+y\}=\mathcal{L}\{1\}$$

$$\mathcal{L}\{y''\}+\mathcal{L}\{y\}=\frac{1}{s}$$

$$s^2\mathcal{L}\{y\}-sy(0)-y'(0)+\mathcal{L}\{y\}=\frac{1}{s}$$

$$\mathcal{L}\{y\}(s^2+1)-2s-3=\frac{1}{s}$$

$$\mathcal{L}\{y\}(s^2+1)=\frac{1}{s}+2s+3$$

$$Y(s)=\mathcal{L}\{y\}=\frac{1+2s^2+3s}{s}\cdot\frac{1}{(s^2+1)}$$

$$=\frac{1+3s+2s^2}{s(s^2+1)}$$

* In 7.5 we'll take inverse transform on both sides to get y. (which is the solution to d.e. $y''+y=1$).

#9. $\mathcal{L}\{\int_0^t e^{-\tau}\cos\tau\,d\tau\}$?

this is not in the form $g(\tau)f(t-\tau)$.

$$\int_0^t e^{-\tau}\cos\tau\,d\tau = \int_0^t e^{-\tau+t-t}\cos\tau\,d\tau$$

$$= e^{-t}\int_0^t e^{t-\tau}\cos\tau\,d\tau$$

(since we integral with respect to τ, we can pull out e^{-t}.)

$$= e^{-t}(e^t * \cos t).$$

So $\mathcal{L}\{\int_0^t e^{-\tau}\cos\tau\,d\tau\} = \mathcal{L}\{e^{-t}(e^t * \cos t)\}$

$$= \mathcal{L}\{e^t * \cos t\}_{s\to s+1}$$

(1st translation)

$$= \mathcal{L}\{e^t\}_{s\to s+1} \cdot \mathcal{L}\{\cos t\}_{s\to s+1}$$

$$= \frac{1}{s-1}\Big|_{s\to s+1} \cdot \frac{s}{s^2+1}\Big|_{s\to s+1}$$

$$= \frac{1}{s+1-1}\cdot\frac{s+1}{(s+1)^2+1} = \frac{1}{s}\cdot\frac{s+1}{(s+1)^2+1}$$

#29. $\boxed{f*g = g*f}$

Let $u = t-\tau$ $\quad du = -d\tau.$ $\quad \tau = t-u$ $\quad \tau=0 \;\; u=t.$

$\tau = t \;\; u = 0.$

$$f*g = \int_0^t f(\tau)\, g(t-\tau)\, d\tau.$$

$$= -\int_t^0 f(t-u)\, g(u)\, du.$$

$$= \int_0^t f(t-u)\, g(u)\, du$$

$$= \int_0^t g(u)\, f(t-u)\, du$$

$$= g*f.$$

#30. $\boxed{f*(g+h) = f*g + f*h}$

$$f*(g+h) = \int_0^t f(\tau)\,[g(t-\tau)+h(t-\tau)]\,d\tau$$

$$= \int_0^t f(\tau)\,g(t-\tau)\,d\tau + \int_0^t f(\tau)\,h(t-\tau)\,d\tau$$

$$= f*g + f*h.$$

#8 $\mathcal{L}\left\{\int_0^t \cos\tau \, d\tau\right\}$

$= \mathcal{L}\left\{\int_0^t \cos\tau \cdot 1 \, d\tau\right\}$ $\quad f(t) = \cos t \quad g(t) = 1.$

$g(t-\tau) = 1.$

$= \mathcal{L}\{f(t)\} \cdot \mathcal{L}\{g(t)\}$

$= \mathcal{L}\{\cos t\} \cdot \mathcal{L}\{1\}$

$= \frac{1}{s} \mathcal{L}\{\cos t\}$

$= \frac{1}{s} \frac{s}{(s^2+1)} = \frac{1}{s^2+1}$

#10 $\mathcal{L}\{\int_0^t \tau \sin\tau \, d\tau\}$.

$f = t\sin t \quad g = 1.$

$g(t-\tau) = 1.$

$= \mathcal{L}\{f(t)\} \cdot \mathcal{L}\{g(t)\}$

$= \mathcal{L}\{t\sin t\} \cdot \mathcal{L}\{1\}$

$= \frac{1}{s} \mathcal{L}\{t\sin t\}$

$= \frac{1}{s}\left(-\frac{d}{ds}\mathcal{L}\{\sin t\}\right)$

$= \frac{1}{s}\left(-\frac{d}{ds}\left(\frac{1}{s^2+1}\right)\right)$

$= \frac{1}{s} \frac{2s}{(s^2+1)^2}$

$= \frac{2}{(s^2+1)^2}.$

\# 12. $\mathcal{L}\{\int_0^t \sin\tau \cos(t-\tau)\,d\tau\}$

$f(t) = \sin t$

$g(t) = \cos t$.

$= \mathcal{L}\{\sin t\}\mathcal{L}\{\cos t\}$

$= \dfrac{1}{s^2+1} \cdot \dfrac{s}{s^2+1}$

$= \dfrac{s}{(s^2+1)^2}$

\#14. $\mathcal{L}\{t\int_0^t \tau e^{-\tau}\,d\tau\} = -\frac{d}{ds}\mathcal{L}\{\int_0^t \tau e^{-\tau}\,d\tau\}$

$= -\dfrac{d}{ds}\left(\dfrac{1}{s}\mathcal{L}\{te^{-t}\}\right)$

$= -\dfrac{d}{ds}\left(\dfrac{1}{s}\left[-\dfrac{d}{ds}\mathcal{L}(e^{-t})\right]\right)$.

$= -\dfrac{d}{ds}\left(\dfrac{1}{s}\left[-\dfrac{d}{ds}\dfrac{1}{s+1}\right]\right)$

$= -\dfrac{d}{ds}\left(\dfrac{1}{s}\dfrac{1}{(s+1)^2}\right) = -\dfrac{-\left[(s+1)^2 + s2(s+1)\right]}{s^2(s+1)^4} = \dfrac{(s+1)+2s}{s^2(s+1)^3} = \dfrac{3s+1}{s^2(s+1)^3}$

#16 $\mathcal{L}\{1 * e^{-2t}\} = \frac{1}{s(s+2)}$

#18 $\mathcal{L}\{t^2 * te^{t}\} = \frac{2}{s^3}\mathcal{L}\{te^{t}\}$

$= \frac{2}{s^3} - \frac{d}{ds}\frac{1}{s-1} = \frac{2}{s^3}\frac{1}{(s-1)^2}$

#20 $\mathcal{L}\{e^{2t} * \sin t\} = \mathcal{L}\{e^{2t}\}\mathcal{L}\{\sin t\}$

$= \frac{1}{s-2} \cdot \frac{1}{s^2+1}$

#22. $\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}F(s)\right\} = \cos 2t * f(t) = \int_0^t f(\tau)\cos 2(t-\tau)\,d\tau.$

#24. $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\cdot\frac{1}{s^2+1}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$

$$= 1 * \sin t = \int_0^t \sin(t-\tau)\,d\tau$$

$$= \cos(t-\tau)\Big|_0^t$$

$$= 1-\cos t.$$

#26. $\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$

$$= e^{-t} * e^{-t}$$

$$= \int_0^t e^{-\tau}e^{-(t-\tau)}\,d\tau = e^{-t}\int_0^t d\tau = te^{-t}.$$

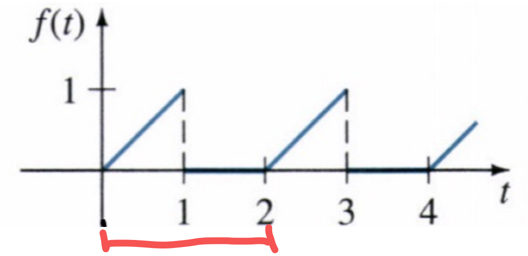
7.4.3

THEOREM 7.10 Transform of a Periodic Function

Let $f(t)$ be piecewise continuous on $[0, \infty)$ and of exponential order. If $f(t)$ is periodic with period T, then

$$\mathscr{L}\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt. \tag{8}$$

e.g.) Compute $\mathscr{L}\{f(t)\}$



This function is periodic with period $T=2$.
Since the value of the function is the same on $[0,2)$, $[2,4)$, $\cdots$

so, $\mathscr{L}\{f(t)\} = \frac{1}{1-e^{-s\cdot 2}} \int_0^2 e^{-st} f(t)\,dt$, on $[0,2)$ $f(t) = \begin{cases} t & 0 \le t < 1 \\ 0 & 1 \le t < 2 \end{cases}$

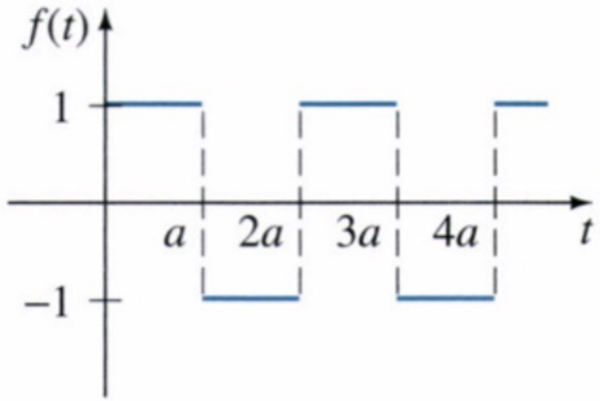
$$= \frac{1}{1-e^{-2s}} \left[\int_0^1 e^{-st} \cdot t\,dt + \int_1^2 e^{-st} \cdot 0\,dt \right]$$

$$= \frac{1}{1-e^{-2s}} \left[\int_0^1 \frac{t}{-s}\, de^{-st} \right]$$

$$= \frac{1}{1-e^{-2s}} \left[\frac{t}{-s} e^{-st} \Big|_0^1 - \int_0^1 \frac{e^{-st}}{(-s)^2} d(-st) \right] = \frac{1}{1-e^{-2s}} \left[-\frac{e^{-s}}{s} - \frac{e^{-s}-1}{s^2} \right]$$

e.g.). Compute $\mathcal{L}\{f(t)\}$

period $T=2a$. with $f(t)=\begin{cases} 1 & 0\le t<a \\ -1 & a\le t<2a \end{cases}$



$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-s\cdot 2a}} \int_0^{2a} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2as}} \left[\int_0^a e^{-st}\,dt + \int_a^{2a} -e^{-st}\,dt \right]$$

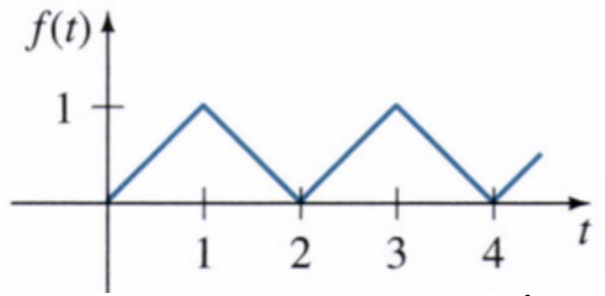
$$= \frac{1}{1-e^{-2as}} \left[\frac{e^{-st}}{-s} \Big|_0^a + \frac{e^{-st}}{s} \Big|_a^{2a} \right]$$

$$= \frac{1}{1-e^{-2as}} \left[\frac{e^{-sa}-1}{-s} + \frac{e^{-2as}-e^{-sa}}{s} \right]$$

$$= \frac{1}{1-e^{-2as}} \frac{e^{-2as}-2e^{-sa}+1}{s}$$

e.g.) compute $\mathcal{L}\{f(t)\}$

period $T=2$ $\quad f(t)=\begin{cases} t & 0\le t<1 \\ 2-t & 1\le t<2 \end{cases}$



$$\mathcal{L}\{f(t)\}=\frac{1}{1-e^{-2s}}\int_0^2 e^{-st}f(t)\,dt.$$

$$=\frac{1}{1-e^{-2s}}\left[\int_0^1 e^{-st}\,t\,dt+\int_1^2 e^{-st}(2-t)\,dt\right]$$

$u=t \quad dv=e^{-st}dt \qquad\qquad u=2-t \quad dv=-e^{-st}dt$

$du=dt \quad v=\frac{e^{-st}}{-s} \qquad\qquad du=-dt \quad v=\frac{e^{-st}}{s}$

$$=\frac{1}{1-e^{-2s}}\left[-\frac{te^{-st}}{s}\Big|_0^1-\int_0^1\frac{e^{-st}}{-s}dt-(2-t)\frac{e^{-st}}{s}\Big|_1^2+\int_1^2\frac{e^{-st}}{s}dt\right.$$

$$=\frac{1}{1-e^{-2s}}\left[-\frac{t}{s}e^{-st}-\frac{1}{s^2}e^{-st}\Big|_0^1+\frac{-(2-t)}{s}e^{-st}\Big|_1^2-\frac{1}{s^2}e^{-st}\Big|_1^2\right.$$

$$=\frac{1}{1-e^{-2s}}\left[\cancel{-\frac{1}{s}e^{-s}}\cancel{-\frac{1}{s^2}e^{-s}}+\frac{1}{s^2}+\cancel{\frac{1}{s}e^{-s}}-\frac{1}{s^2}e^{-2s}\cancel{+\frac{1}{s^2}e^{-s}}\right]$$

$$=\frac{1}{1-e^{-2s}}\left[\frac{1-e^{-2s}}{s^2}\right]=\frac{1}{s^2}.$$

7.3.1

THEOREM 7.5 **First Translation Theorem**

If a is any real number, then

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a),$$

where $F(s) = \mathcal{L}\{f(t)\}$.

THEOREM 7.2 **Transforms of Some Basic Functions**

(a) $\mathcal{L}\{1\} = \dfrac{1}{s}$

(b) $\mathcal{L}\{t^n\} = \dfrac{n!}{s^{n+1}}, \quad n = 1, 2, 3, \ldots$

(c) $\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}$

(d) $\mathcal{L}\{\sin kt\} = \dfrac{k}{s^2+k^2}$

(e) $\mathcal{L}\{\cos kt\} = \dfrac{s}{s^2+k^2}$

(f) $\mathcal{L}\{\sinh kt\} = \dfrac{k}{s^2-k^2}$

(g) $\mathcal{L}\{\cosh kt\} = \dfrac{s}{s^2-k^2}$

e.g.) $\mathcal{L}\{e^{5t}t^3\} = \mathcal{L}\{t^3\}\Big|_{s\to s-5}$

$$= \frac{3!}{s^{3+1}}\Bigg|_{s\to s-5}$$

$$= \frac{3!}{(s-5)^4}$$

these are reminder replace s with s-5 at the last step

e.g.) $\mathcal{L}\{e^{-2t}\cos 2t\} = \mathcal{L}\{\cos 2t\}\Big|_{s\to s+2}$

$$= \frac{s}{s^2+2^2}\Bigg|_{s\to s+2} = \frac{s+2}{(s+2)^2+4}$$

e.g.) $\mathcal{L}\{e^{-3t}e^{4t}\} = \mathcal{L}\{e^{4t}\}\Big|_{s\to s+3} = \dfrac{1}{s-4}\Bigg|_{s\to s+3} = \dfrac{1}{s+3-4} = \dfrac{1}{s-1}$

$$= \mathcal{L}\{e^{t}\} = \frac{1}{s-1}$$

(* sometimes you don't have to use 1st translation theorem) e.g.) $\mathcal{L}\{te^{-6t}\}$

$= \mathcal{L}\{e^{-6t}t\}$

$= \mathcal{L}\{t\}\Big|_{s\to s+6}$

$= \frac{1}{s^2}\Big|_{s\to s+6}$

$= \frac{1}{(s+6)^2}$

e.g.) $\mathcal{L}\{t^{10}e^{-7t}\}$

$= \mathcal{L}\{e^{-7t}t^{10}\}$

$= \frac{10!}{s^{11}}\Big|_{s\to s+7}$

$= \frac{10!}{(s+7)^{11}}$

e.g.) $\mathcal{L}\{e^{-2t}\cos 4t\}$

$k=4$

$= \frac{s}{s^2+4^2}\Big|_{s\to s+2}$

$= \frac{s+2}{(s+2)^2+16}$

e.g) $\mathcal{L}\{e^{-t}\cosh t\}$

$$= \frac{s}{s^2-1^2}\Big|_{s\to s-(-1)}$$

$$= \frac{s}{s^2-1}\Big|_{s\to s+1}$$

$$= \frac{s+1}{(s+1)^2-1}$$

e.g) $\mathcal{L}\{e^{2t}(t-1)^2\}$

$$=\mathcal{L}\{e^{2t}(t^2-2t+1)\}$$

$$=\mathcal{L}\{e^{2t}t^2\}-2\mathcal{L}\{e^{2t}t\}+\mathcal{L}\{e^{2t}\}$$

$$= \frac{2!}{s^3}\Big|_{s\to s-2} - 2\frac{1}{s^2}\Big|_{s\to s-2}+\frac{1}{s}\Big|_{s\to s-2}$$

$$= \frac{2}{(s-2)^3}-\frac{2}{(s-2)^2}+\frac{1}{s-2}$$

e.g) $\mathcal{L}\{t(e^{t}+e^{2t})^2\}$

$$=\mathcal{L}\{te^{2t}+2te^{3t}+te^{4t}\}$$

$$=\mathcal{L}\{e^{2t}t\}+2\mathcal{L}\{e^{3t}t\}+\mathcal{L}\{e^{4t}t\}$$

$$=\frac{1}{s^2}\Big|_{s\to s-2}+\frac{2}{s^2}\Big|_{s\to s-3}+\frac{1}{s^2}\Big|_{s\to s-4}$$

$$=\frac{1}{(s-2)^2}+\frac{2}{(s-3)^2}+\frac{1}{(s-4)^2}$$

e.g.)

$\mathcal{L}\{e^{-t}\sin^2 t\}$

$= \mathcal{L}\{e^{-t}\frac{1-\cos 2t}{2}\}$

$= \frac{1}{2}\mathcal{L}\{e^{-t}\} - \frac{1}{2}\mathcal{L}\{e^{-t}\cos 2t\}$

$= \frac{1}{2}\frac{1}{s}\Big|_{s\to s+1} - \frac{1}{2}\frac{s}{s^2+4}\Big|_{s\to s+1}$

$= \frac{1}{2}\frac{1}{s+1} - \frac{1}{2}\frac{s+1}{(s+1)^2+4}$

Inverse Form of the First Translation Theorem The inverse form of Theorem 7.5 can be written

$$\mathscr{L}^{-1}\{F(s-a)\} = \mathscr{L}^{-1}\{F(s)|_{s\to s-a}\} = e^{at}f(t), \tag{1}$$

where $f(t) = \mathscr{L}^{-1}\{F(s)\}$.

eg) $\mathscr{L}^{-1}\left\{\frac{s+3}{(s+3)^2+1}\right\} = \mathscr{L}^{-1}\left\{\frac{s}{s^2+1}\Big|_{s\to s+3}\right\} = e^{-3t}\cos t.$

eg) $\mathscr{L}^{-1}\left\{\frac{3!}{(s-1)^4}\right\} = \mathscr{L}^{-1}\left\{\frac{3!}{s^4}\Big|_{s\to s-1}\right\} = e^{t}t^{3}$

eg) $\mathscr{L}^{-1}\left\{\frac{s-3}{s^2-6s+11}\right\} = \mathscr{L}^{-1}\left\{\frac{s-3}{(s-3)^2+2}\right\} = \mathscr{L}^{-1}\left\{\frac{s}{s^2+2}\Big|_{s\to s-3}\right\} = e^{3t}\cos\sqrt{2}t$

*

* In both cases, we can use partial fraction decomposition But that takes too much time! avoid using partial fraction decomposition if possible!

* $s^2-6s+11 = s^2-6s+9+2 = (s-3)^2+2$ (use completing the square)

eg) $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+2s+5}\right\}$

$= \mathcal{L}^{-1}\left\{\dfrac{1}{(s+1)^2+4}\right\}$

$= \mathcal{L}^{-1}\left\{\dfrac{1}{2}\dfrac{2}{(s+1)^2+2^2}\right\}$

$= \dfrac{1}{2}\mathcal{L}^{-1}\left\{\dfrac{2}{(s+1)^2+2^2}\right\}.$

$= \dfrac{1}{2}\mathcal{L}^{-1}\left\{\dfrac{2}{s^2+2^2}\Big|_{s\to s+1}\right\}.$

$= \dfrac{1}{2}e^{-t}\sin 2t$ $\qquad a=-1.$

$k=2.$

THEOREM 7.3 Some Inverse Transforms

(a) $1 = \mathscr{L}^{-1}\left\{\dfrac{1}{s}\right\}$

(b) $t^n = \mathscr{L}^{-1}\left\{\dfrac{n!}{s^{n+1}}\right\}, n = 1, 2, 3, \ldots$ (c) $e^{at} = \mathscr{L}^{-1}\left\{\dfrac{1}{s-a}\right\}$

(d) $\sin kt = \mathscr{L}^{-1}\left\{\dfrac{k}{s^2+k^2}\right\}$ ☆ (e) $\cos kt = \mathscr{L}^{-1}\left\{\dfrac{s}{s^2+k^2}\right\}$

(f) $\sinh kt = \mathscr{L}^{-1}\left\{\dfrac{k}{s^2-k^2}\right\}$ (g) $\cosh kt = \mathscr{L}^{-1}\left\{\dfrac{s}{s^2-k^2}\right\}$

$F(s) = \dfrac{2}{s^2+2^2}$

$\mathcal{L}^{-1}\{F(s)\} = \sin 2t.$

$s^2+2s+5 = s^2+2s\boxed{+1}+\cancel{4}$

$= (s+1)^2+\cancel{4}$

e.g) $\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^4}\right\}$.

$n=3$.

$n!=3!$

$= \mathcal{L}^{-1}\left\{\frac{1}{3!}\frac{3!}{(s-1)^{3+1}}\right\}$

$= \frac{1}{3!}\mathcal{L}^{-1}\left\{\frac{3!}{(s-1)^{3+1}}\right\}$.

$= \frac{1}{3!}\mathcal{L}^{-1}\left\{\frac{3!}{s^{3+1}}\Big|_{s\to s-1}\right\}$

$F(s) = \frac{3!}{s^{3+1}}$

$\mathcal{L}^{-1}\{F(s)\} = t^3 = f(t)$

$a=1$

$= \frac{1}{3!}e^t \cdot t^3$.

$= \frac{1}{6}t^3e^t$.

Inverse Form of the First Translation Theorem The inverse form of Theorem 7.5 can be written

$$\mathcal{L}^{-1}\{F(s-a)\} = \mathcal{L}^{-1}\{F(s)|_{s\to s-a}\} = e^{at}f(t), \qquad (1)$$

where $f(t) = \mathcal{L}^{-1}\{F(s)\}$.

$a=1$

THEOREM 7.3 **Some Inverse Transforms**

(a) $1 = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}$

(b) $t^n = \mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\}, n = 1, 2, 3, \ldots$ (c) $e^{at} = \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\}$

(d) $\sin kt = \mathcal{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\}$ (e) $\cos kt = \mathcal{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\}$

(f) $\sinh kt = \mathcal{L}^{-1}\left\{\frac{k}{s^2-k^2}\right\}$ (g) $\cosh kt = \mathcal{L}^{-1}\left\{\frac{s}{s^2-k^2}\right\}$

e.g) $\mathcal{L}^{-1}\left\{\frac{5s}{(s-2)^2}\right\}$.

$= \mathcal{L}^{-1}\left\{\frac{5(s-2)+10}{(s-2)^2}\right\}$.

$= \mathcal{L}^{-1}\left\{\frac{5}{s-2} + \frac{10}{(s-2)^2}\right\}$.

Linear.

$= 5\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + 10\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2}\right\}$

$= 5\mathcal{L}^{-1}\left\{\frac{1}{s}\Big|_{s\to s-2}\right\} + 10\mathcal{L}^{-1}\left\{\frac{1}{s^2}\Big|_{s\to s-2}\right\}$

$a=2$ $\quad a=2.$

$= 5e^{2t}\cdot 1 + 10e^{2t}\cdot t.$

$= 5e^{2t} + 10te^{2t}.$

Inverse Form of the First Translation Theorem The inverse form of Theorem 7.5 can be written

$$\mathscr{L}^{-1}\{F(s-a)\} = \mathscr{L}^{-1}\{F(s)|_{s\to s-a}\} = e^{at}f(t), \tag{1}$$

where $f(t) = \mathscr{L}^{-1}\{F(s)\}$.

THEOREM 7.3 Some Inverse Transforms

(a) $1 = \mathscr{L}^{-1}\left\{\frac{1}{s}\right\}$

(b) $t^n = \mathscr{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\}, n = 1, 2, 3, \ldots$ (c) $e^{at} = \mathscr{L}^{-1}\left\{\frac{1}{s-a}\right\}$

(d) $\sin kt = \mathscr{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\}$ (e) $\cos kt = \mathscr{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\}$

(f) $\sinh kt = \mathscr{L}^{-1}\left\{\frac{k}{s^2-k^2}\right\}$ (g) $\cosh kt = \mathscr{L}^{-1}\left\{\frac{s}{s^2-k^2}\right\}$

$F_1(s) = \frac{1}{s}$

$\mathcal{L}^{-1}\{F_1(s)\} = 1$

$F_2(s) = \frac{1}{s^2} = \frac{1}{s^{1+1}} = \frac{1!}{s^{1+1}}$

$\mathcal{L}^{-1}\{F_2(s)\} = t^1 = t.$

e.g

$$\mathcal{L}^{-1}\left\{\frac{2s+5}{s^2+6s+34}\right\}.$$

#47
#6.1
7.1-7.3.

Inverse Form of the First Translation Theorem The inverse form of Theorem 7.5 can be written

$$\mathcal{L}^{-1}\{F(s-a)\} = \mathcal{L}^{-1}\{F(s)|_{s\to s-a}\} = e^{at}f(t), \tag{1}$$

where $f(t) = \mathcal{L}^{-1}\{F(s)\}$.

$\mathcal{L}\{e^{at}f(t)\}$. $\boxed{\mathcal{L}\{t^n f(t)\}}$

THEOREM 7.3 Some Inverse Transforms

(a) $1 = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}$

(b) $t^n = \mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\}, n = 1, 2, 3, \ldots$ (c) $e^{at} = \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\}$

(d) $\sin kt = \mathcal{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\}$ (e) $\cos kt = \mathcal{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\}$

(f) $\sinh kt = \mathcal{L}^{-1}\left\{\frac{k}{s^2-k^2}\right\}$ (g) $\cosh kt = \mathcal{L}^{-1}\left\{\frac{s}{s^2-k^2}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{2s+5}{(s+3)^2+25}\right\}.$$

$$= \mathcal{L}^{-1}\left\{\frac{2(s+3)-1}{(s+3)^2+25}\right\}.$$

$$= \mathcal{L}^{-1}\left\{\frac{2(s+3)}{(s+3)^2+25}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s+3)^2+25}\right\}$$

$$= 2\mathcal{L}^{-1}\left\{\frac{(s+3)}{(s+3)^2+25}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s+3)^2+25}\right\}.$$

$$\boxed{\begin{aligned} &s^2+6s+9+25. \\ &= (s+3)^2+25. \end{aligned}}$$

$$= 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+5^2}\Big|_{s\to s+3}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{5}\frac{5}{s^2+5^2}\Big|_{s\to s+3}\right\}.$$

$a=-3$

$$= 2e^{-3t}\cdot\cos 5t - e^{-3t}\cdot\frac{1}{5}\sin 5t.$$

$a=-3$ $F_1(s) = \frac{s}{s^2+5^2}$ $F_2(s) = \frac{5}{s^2+5^2}\cdot\frac{1}{5}$

$\mathcal{L}^{-1}\{F_1(s)\} = \cos 5t$ $\mathcal{L}^{-1}\{F_2(s)\} = \frac{1}{5}\cdot\sin 5t$

7.4.1

THEOREM 7.7 **Derivatives of Transforms**

For $n = 1, 2, 3, \ldots,$

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s),$$

where $F(s) = \mathscr{L}\{f(t)\}$.

* There is no inverse form for this theorem

e.g) $\mathscr{L}\{te^{3t}\} = (-1)^1 \frac{d}{ds}\left(\mathscr{L}\{e^{3t}\}\right)$

$$= (-1)\frac{d}{ds}\left(\frac{1}{s-3}\right) = (-1)\frac{d}{ds}(s-3)^{-1}$$

$$= (-1)(-1)(s-3)^{-2} \cdot 1 = (s-3)^{-2}$$

e.g) $\mathscr{L}\{te^{-t}\cos t\} = (-1)^1 \frac{d}{ds}\left(\mathscr{L}\{e^{-t}\cos t\}\right)$ • derivative

$$= (-1)\frac{d}{ds}\left(\mathscr{L}\{\cos t\}_{s\to s-(-1)}\right)$$ • 1st translation

$$= (-1)\frac{d}{ds}\left(\frac{s}{s^2+1}\Big|_{s\to s+1}\right)$$

$$= (-1)\frac{d}{ds}\left(\frac{s+1}{(s+1)^2+1}\right) = (-1)\frac{(s+1)^2+1-(s+1)2(s+1)}{((s+1)^2+1)^2}$$

$$= \frac{(s+1)^2-1}{((s+1)^2+1)^2}$$

eg) $\mathcal{L}\{t^2e^{3t}\} = (-1)^2 \dfrac{d^2}{ds^2}\left(\dfrac{1}{s-3}\right)$

※ take derivative twice

$$= \frac{d}{ds}\left(-1(s-3)^{-2}\right)$$

$$= (-1)(-2)(s-3)^{-3}$$

$$= \frac{2}{(s-3)^3}$$

e.g).

$\mathcal{L}\{t\cos 2t\}$

$= (-1)\frac{d}{ds}\left(\frac{s}{s^2+4}\right)$

$= (-1)\frac{s^2+4-s\cdot 2s}{(s^2+4)^2}$

$= \frac{s^2-4}{(s^2+4)^2}$

e.g).

$\mathcal{L}\{t\sinh 3t\}$

$= -\frac{d}{ds}\left(\frac{3}{s^2-9}\right)$

$= \frac{6s}{(s^2-9)^2}$

e.g).

$\mathcal{L}\{t^2\cos t\}$

$= \frac{d}{ds^2}\left(\frac{s}{s^2+1}\right)$

$= \frac{d}{ds}\left(\frac{1-s^2}{(s^2+1)^2}\right)$

$= \frac{-2s(s^2+1)^2-2(1-s^2)(s^2+1)s\cdot 2}{(s^2+1)^4}$

$= \frac{-2s(s^2+1)-4s(1-s^2)}{(s^2+1)^3}$

$= \frac{-2s^3-2s-4s+4s^3}{(s^2+1)^3}$

$= \frac{2s^3-6s}{(s^2+1)^3} = \frac{2s(s^2-3)}{(s^2+1)^3}$

Use Thm 7.7. in the form $(n=1)$

$$f(t) = -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{d}{ds}F(s)\right\}.$$

to evaluate the given inverse Laplace transform.

$\boxed{62}$ $f(t) = \mathcal{L}^{-1}\left\{\ln\frac{s^2+1}{s^2+4}\right\} = -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{d}{ds}\left[\ln(s^2+1) - \ln(s^2+4)\right]\right\}$

$$= -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{2s}{s^2+1} - \frac{2s}{s^2+4}\right\}$$

$$= -\frac{1}{t}\left[\mathcal{L}^{-1}\left\{\frac{2s}{s^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{2s}{s^2+4}\right\}\right]$$

$$= -\frac{1}{t}\left[2\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} - 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}\right]$$

$$= -\frac{1}{t}\left[2\cos t - 2\cos 2t\right]$$

$\boxed{64}$. $f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s} - \cot^{-1}\frac{4}{s}\right\}$

$= -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{d}{ds}\left(\frac{1}{s} - \cot^{-1}\frac{4}{s}\right)\right\}$

$= -\frac{1}{t}\mathcal{L}^{-1}\left\{-\frac{1}{s^2} - \frac{4}{s^2+4^2}\right\}$

$= -\frac{1}{t}\left[-\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{4}{s^2+4^2}\right\}\right]$

$= -\frac{1}{t}\left[-t - \sin 4t\right]$

$= 1 + \frac{1}{t}\sin 4t$

$f(t) = -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{d}{ds}F(s)\right\}$.

$\frac{d}{ds}\cot^{-1}s = -\frac{1}{1+s^2}$

$\frac{d}{ds}\cot^{-1}\frac{4}{s} = -\frac{1}{1+\left(\frac{4}{s}\right)^2} \cdot -\frac{4}{s^2}$

$= \frac{4}{1+\left(\frac{4}{s}\right)^2} \cdot \frac{1}{s^2}$

$= \frac{4}{s^2+4^2}$

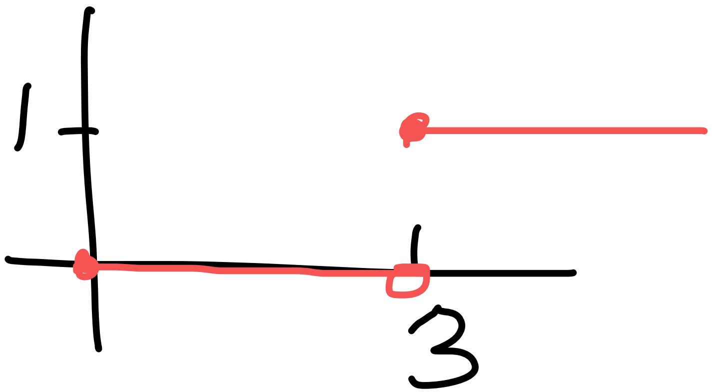
DEFINITION 7.3 Unit Step Function

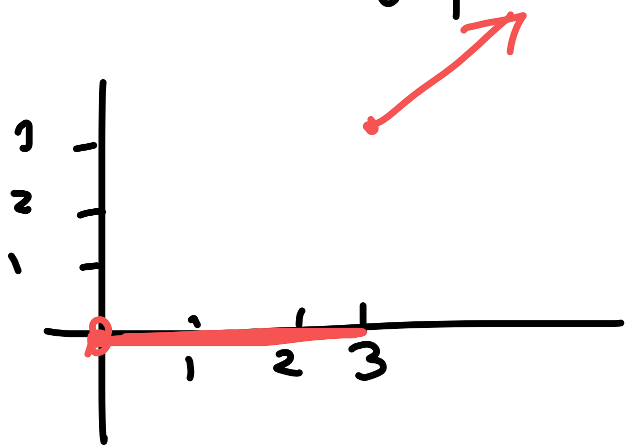
The function $\mathcal{U}(t-a)$ is defined to be

$$\mathcal{U}(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a \end{cases}$$

e.g) $\mathcal{U}(t-3) = \begin{cases} 0 & 0 \le t < 3 \\ 1 & t \ge 3 \end{cases}$

$t\,\mathcal{U}(t-3) = \begin{cases} t \; 0 & 0 \le t < 3 \\ t \; 1 & t \ge 3 \end{cases}$





The unit step function can also be used to write piecewise-defined functions in a compact form. For instance, the piecewise-defined function

$$f(t) = \begin{cases} g(t), & 0 \le t < a \\ h(t), & t \ge a \end{cases} \tag{2}$$

can be written as

$$f(t) = g(t) - g(t)\mathcal{U}(t-a) + h(t)\mathcal{U}(t-a). \tag{3}$$

Similarly, a function of the type

$$f(t) = \begin{cases} 0, & 0 \le t < a \\ g(t), & a \le t < b \\ 0, & t \ge b \end{cases} \tag{4}$$

can be written

$$f(t) = g(t)[\mathcal{U}(t-a) - \mathcal{U}(t-b)]. \tag{5}$$

eg) Express f(t) in terms of a unit function

$$f(t) = \begin{cases} 20t & 0 \le t < 5 \\ 0 & t \ge 5 \end{cases} \rightarrow f(t) = 20t - 20t\,\mathcal{U}(t-5) + 0\,\mathcal{U}(t-5)$$

$$f(t) = \begin{cases} 0 & 0 \le t < 4 \\ t & 4 \le t < 5 \\ 0 & t \ge 5 \end{cases} \longrightarrow f(t) = t[\mathcal{U}(t-4) - \mathcal{U}(t-5)]$$

$$f(t) = \begin{cases} t & 0 \le t < 4 \\ 0 & 4 \le t < 5 \\ t & t \ge 5 \end{cases} \longrightarrow f(t) = t - t[\mathcal{U}(t-4) - \mathcal{U}(t-5)]$$

think of this as

$$t - \begin{cases} 0 & 0 \le t < 4 \\ t & 4 \le t < 5 \\ 0 & t \ge 5 \end{cases}$$

7.3.2

THEOREM 7.6 **Second Translation Theorem**

If a is a positive constant, then

$$\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\} = e^{-as}F(s),$$

where $F(s) = \mathscr{L}\{f(t)\}$.

We often wish to find the Laplace transform of just the unit step function. This can be found from either Definition 7.1 or Theorem 7.6. If we identify $f(t) = 1$ in Theorem 7.6, then $f(t-a) = 1$, $F(s) = \mathscr{L}\{1\} = 1/s$, and so

$$\mathscr{L}\{\mathscr{U}(t-a)\} = \frac{e^{-as}}{s}. \tag{7}$$

e.g) $\mathscr{L}\{(t-2)^3 U(t-2)\} = e^{-2s}\mathscr{L}\{t^3\} = e^{-2s}\frac{3!}{s^{3+1}} = \frac{3!\,e^{-2s}}{s^4}$

e.g) $\mathscr{L}\{f(t)\}$ where $f(t) = \begin{cases} 20 & 0 \le t < 5 \\ 0 & t \ge 5 \end{cases}$

Now we don't have to use the definition

$f(t) = 20 - 20U(t-5) + 0U(t-5) = 20 - 20U(t-5)$

So $\mathscr{L}\{f(t)\} = \mathscr{L}\{20 - 20U(t-5)\}$

$= \mathscr{L}\{20\} - \mathscr{L}\{20U(t-5)\}$

$= 20\mathscr{L}\{1\} - 20\mathscr{L}\{U(t-5)\}$

$= \frac{20}{s} - 20\frac{e^{-5s}}{s}$

e.g) $\mathcal{L}\{(2t-3)\,u(t-1)\}$

$= \mathcal{L}\{(2(t-1)-1)\,u(t-1)\}$

$= e^{-s}\mathcal{L}\{2t-1\}$

$= e^{-s}[2\mathcal{L}\{t\} - \mathcal{L}\{1\}]$

$= e^{-s}[2\frac{1}{s^2} - \frac{1}{s}]$

* We need to have $(t-1)$ to use 2nd translation thm

$f(t) = 2t-1$

e.g). $\mathcal{L}\{e^{2-t}\mathcal{U}(t-2)\}$

$= \mathcal{L}\{e^{-(t-2)}\mathcal{U}(t-2)\}$. $\quad a=2$

$= e^{-2s}\mathcal{L}\{e^{-t}\}$ $\quad f(t)=e^{-t}$

$= \dfrac{e^{-2s}}{s-(-1)}$

$= \dfrac{e^{-2s}}{s+1}$

e.g) $\mathcal{L}\{(3t+1)\mathcal{U}(t-3)\}$

* we need to have $(t-3)$ to use 2nd translation.

$$= 3\mathcal{L}\{(t+\tfrac{1}{3})\mathcal{U}(t-3)\}$$

$$= 3\mathcal{L}\{(t-3+\tfrac{10}{3})\mathcal{U}(t-3)\}$$

$$= 3\mathcal{L}\{(t-3)\mathcal{U}(t-3) + \frac{10}{3}\mathcal{U}(t-3)\}.$$

$a=3$.

$f(t)=t$.

$f_2(t)=1$

$$= 3\left[\frac{e^{-3s}}{s^{1+1}} + \frac{\frac{10}{3}}{s}e^{-3s}\right]$$

$$= \frac{3e^{-3s}}{s^2} + \frac{10e^{-3s}}{s}$$

e.g.) $\mathcal{L}\{\sin t\, \mathcal{U}(t-\frac{\pi}{2})\}$

$= \mathcal{L}\{\cos(t-\frac{\pi}{2})\, \mathcal{U}(t-\frac{\pi}{2})\}$

$= e^{-\frac{\pi}{2}s}\mathcal{L}\{\cos t\}$.

$= \frac{s e^{-\frac{\pi}{2}s}}{s^2+1}$

* we need to have $t-\frac{\pi}{2}$ to use 2n translation.

$a = \frac{\pi}{2}$.

$f(t) = \cos t$.

e.g.) $\mathcal{L}\{te^{t-5}\mathcal{U}(t-5)\}$

$= \mathcal{L}\{(t-5)e^{t-5}\mathcal{U}(t-5) + 5e^{t-5}\mathcal{U}(t-5)\}$

$a = 5$

$= e^{-5s}\mathcal{L}\{te^{t}\} + e^{-5s}\mathcal{L}\{5e^{t}\}$

$f_1(t) = te^{t}.$

$\mathcal{L}\{f_1(t)\} = \mathcal{L}\{te^{t}\}$

$= \dfrac{e^{-5s}}{(s-1)^2} + \dfrac{5e^{-5s}}{(s-1)}$

$= (-1)\dfrac{d}{ds}\mathcal{L}\{e^{t}\}$

$= -1\dfrac{d}{ds}\dfrac{1}{s-1}$

$= -1\dfrac{-1}{(s-1)^2}$

$= \dfrac{1}{(s-1)^2}$

$f_2(t) = e^{t}$

$5\mathcal{L}\{e^{t}\} = \dfrac{5}{s-1}$

Inverse Form of the Second Translation Theorem The inverse form of Theorem 7.6 is

$$\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathscr{U}(t-a), \qquad (8)$$

where $a > 0$ and $f(t) = \mathscr{L}^{-1}\{F(s)\}$.

eg) $\mathscr{L}^{-1}\left\{\frac{e^{-\pi s/2}}{s^2+9}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s^2+9}\right\}_{t\to t-\frac{\pi}{2}} \mathscr{U}(t-\frac{\pi}{2})$

$= \frac{1}{3}\mathscr{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\}_{t\to t-\frac{\pi}{2}} \mathscr{U}(t-\frac{\pi}{2})$

$= \frac{1}{3}\sin(3t)_{t\to t-\frac{\pi}{2}} \mathscr{U}(t-\frac{\pi}{2})$

$= \frac{1}{3}\sin(3(t-\frac{\pi}{2}))\mathscr{U}(t-\frac{\pi}{2})$

eg) $\mathscr{L}^{-1}\left\{\frac{e^{-2s}}{s^3}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s^3}\right\}_{t\to t-2} \mathscr{U}(t-2)$

$= \frac{1}{2!}\mathscr{L}^{-1}\left\{\frac{2!}{s^{2+1}}\right\}_{t\to t-2} \mathscr{U}(t-2)$

$= \frac{1}{2} t^2\Big|_{t\to t-2} \mathscr{U}(t-2)$

$= \frac{1}{2}(t-2)^2\mathscr{U}(t-2)$

e.g.) $\mathcal{L}^{-1}\left\{\frac{(1+e^{-2s})^2}{s+2}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s+2} + \frac{2e^{-2s}}{s+2} + \frac{e^{-4s}}{s+2}\right\}$$

$a=2$ $\quad$ $a=4$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}\Big|_{t\to t-2}\mathcal{U}(t-2) + \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}\Big|_{t\to t-4}\mathcal{U}(t-4)$$

$$= e^{-2t} + 2\, e^{-2t}\Big|_{t\to t-2}\mathcal{U}(t-2) + e^{-2t}\Big|_{t\to t-4}\mathcal{U}(t-4)$$

$$= e^{-2t} + 2e^{-2(t-2)}\mathcal{U}(t-2) + e^{-2(t-4)}\mathcal{U}(t-4)$$

34. $\mathcal{L}^{-1}\left\{\dfrac{se^{-\pi s/2}}{s^2+4}\right\}$

$a = \frac{\pi}{2}.$

$$= \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}\Big|_{t \to t-\frac{\pi}{2}} \mathcal{U}\left(t-\frac{\pi}{2}\right)$$

$$= \cos 2t\Big|_{t\to t-\frac{\pi}{2}} \mathcal{U}\left(t-\frac{\pi}{2}\right)$$

$$= \cos 2\left(t-\frac{\pi}{2}\right)\mathcal{U}\left(t-\frac{\pi}{2}\right).$$

36. $\mathcal{L}^{-1}\left\{\dfrac{e^{-2s}}{s^2(s-1)}\right\}$

$(As+B)(s-1)+Cs^2=1.$

$As^2+(B-A)s-B=1.$

$$=\mathcal{L}^{-1}\left\{e^{-2s}\left(\frac{As+B}{s^2}+\frac{C}{(s-1)}\right)\right\}.$$

$-B=1.$

$B=-1$

$$=\mathcal{L}^{-1}\left\{e^{-2s}\left(-\frac{1}{s}-\frac{1}{s^2}+\frac{1}{s-1}\right)\right\}$$

$A=B=-1.$

$$=\mathcal{L}^{-1}\left\{-\frac{e^{-2s}}{s}\right\}+\mathcal{L}^{-1}\left\{-\frac{e^{-2s}}{s^2}\right\}+\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s-1}\right\}$$

$\dfrac{-s-1}{s^2}+\dfrac{1}{s-1}$

$=-\dfrac{1}{s}-\dfrac{1}{s^2}+\dfrac{1}{s-1}.$

$$=-\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}\Big|_{t\to t-2}\mathcal{U}(t-2)-\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}\Big|_{t\to t-2}\mathcal{U}(t-2)+\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}\Big|_{t\to t-2}\mathcal{U}(t-2)$$

$$=-\mathcal{U}(t-2)-(t-2)\mathcal{U}(t-2)+e^{(t-2)}\mathcal{U}(t-2)$$

THEOREM 7.7 **Derivatives of Transforms**

For $n = 1, 2, 3, \ldots$,

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s),$$

where $F(s) = \mathscr{L}\{f(t)\}$.

※ There is no inverse form for this theorem

e.g) $\mathscr{L}\{te^{3t}\} = (-1)^1 \frac{d}{ds}(\mathscr{L}\{e^{3t}\})$

$= (-1)\frac{d}{ds}\left(\frac{1}{s-3}\right) = (-1)\frac{d}{ds}(s-3)^{-1}$

$= (-1)(-1)(s-3)^{-2} \cdot 1 = (s-3)^{-2}$

e.g) $\mathscr{L}\{te^{-t}\cos t\} = (-1)^1 \frac{d}{ds}(\mathscr{L}\{e^{-t}\cos t\})$ • derivative

$= (-1)\frac{d}{ds}\left(\mathscr{L}\{\cos t\}_{s \to s-(-1)}\right)$ • 1st translation

$= (-1)\frac{d}{ds}\left(\frac{s}{s^2+1}\Big|_{s \to s+1}\right)$

$= (-1)\frac{d}{ds}\left(\frac{s+1}{(s+1)^2+1}\right) = (-1)\frac{(s+1)^2+1-(s+1)2(s+1)}{((s+1)^2+1)^2}$

$= \frac{(s+1)^2-1}{((s+1)^2+1)^2}$

eg) $\mathcal{L}\{t^2 e^{3t}\} = (-1)^2 \frac{d^2}{ds^2}\left(\frac{1}{s-3}\right)$

※ take derivative twice

$$= \frac{d}{ds}\left(-1(s-3)^{-2}\right)$$

$$= (-1)(-2)(s-3)^{-3}$$

$$= \frac{2}{(s-3)^3}$$

e.g).

$$\mathcal{L}\{t\cos 2t\}$$

$$= (-1)\frac{d}{ds}\left(\frac{s}{s^2+4}\right)$$

$$= (-1)\frac{s^2+4-s\cdot 2s}{(s^2+4)^2}$$

$$= \frac{s^2-4}{(s^2+4)^2}$$

e.g).

$$\mathcal{L}\{t\sinh 3t\}$$

$$= -\frac{d}{ds}\left(\frac{3}{s^2-9}\right)$$

$$= \frac{6s}{(s^2-9)^2}$$

e.g).

$$\mathcal{L}\{t^2\cos t\}$$

$$= \frac{d}{ds^2}\left(\frac{s}{s^2+1}\right)$$

$$= \frac{d}{ds}\left(\frac{1-s^2}{(s^2+1)^2}\right)$$

$$= \frac{-2s(s^2+1)^2 - 2(1-s^2)(s^2+1)s\cdot 2}{(s^2+1)^4}$$

$$= \frac{-2s(s^2+1)-4s(1-s^2)}{(s^2+1)^3}$$

$$= \frac{-2s^3-2s-4s+4s^3}{(s^2+1)^3}$$

$$= \frac{2s^3-6s}{(s^2+1)^3} = \frac{2s(s^2-3)}{(s^2+1)^3}$$

Use Thm 7.7. in the form $(n=1)$

$$f(t) = -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{d}{ds}F(s)\right\}.$$

to evaluate the given inverse Laplace transform.

$\boxed{62}$ $f(t)=\mathcal{L}^{-1}\left\{\ln\frac{s^2+1}{s^2+4}\right\} = -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{d}{ds}\left[\ln(s^2+1)-\ln(s^2+4)\right]\right\}$

$$= -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{2s}{s^2+1}-\frac{2s}{s^2+4}\right\}$$

$$= -\frac{1}{t}\left[\mathcal{L}^{-1}\left\{\frac{2s}{s^2+1}\right\}-\mathcal{L}^{-1}\left\{\frac{2s}{s^2+4}\right\}\right]$$

$$= -\frac{1}{t}\left[2\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}-2\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}\right]$$

$$= -\frac{1}{t}\left[2\cos t-2\cos 2t\right]$$

$\boxed{64}$. $f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s} - \cot^{-1}\frac{4}{s}\right\}$

$$= -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{d}{ds}\left(\frac{1}{s} - \cot^{-1}\frac{4}{s}\right)\right\}$$

$$= -\frac{1}{t}\mathcal{L}^{-1}\left\{-\frac{1}{s^2} - \frac{4}{s^2+4^2}\right\}$$

$$= -\frac{1}{t}\left[-\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{4}{s^2+4^2}\right\}\right]$$

$$= -\frac{1}{t}\left[-t - \sin 4t\right]$$

$$= 1 + \frac{1}{t}\sin 4t$$

$$\boxed{f(t) = -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{d}{ds}F(s)\right\}}.$$

$$\frac{d}{ds}\cot^{-1}s = -\frac{1}{1+s^2}$$

$$\frac{d}{ds}\cot^{-1}\frac{4}{s} = -\frac{1}{1+\left(\frac{4}{s}\right)^2} \cdot -\frac{4}{s^2}$$

$$= \frac{4}{1+\left(\frac{4}{s}\right)^2} \cdot \frac{1}{s^2}$$

$$= \frac{4}{s^2+4^2}$$

Convolution If functions f and g are piecewise continuous on $[0, \infty)$, then the **convolution** of f and g, denoted by $f * g$, is given by the integral

$$f * g = \int_0^t f(\tau)g(t-\tau)\,d\tau.$$

$f*g=g*f$

e.g.) $t * \sin t = \int_0^t \tau \sin(t-\tau)\,d\tau$

$u = \tau \qquad dv = \sin(t-\tau)\,d\tau$

$du = d\tau \qquad v = \cos(t-\tau).$

$= \tau\cos(t-\tau)\Big|_0^t - \int_0^t \cos(t-\tau)\,d\tau$

$= t\cos 0 - 0 - (-\sin(t-\tau))\Big|_0^t$

$= t - \sin(0) - (+\sin t)$

$= t - \sin t.$

THEOREM 7.9 **Convolution Theorem**

Let $f(t)$ and $g(t)$ be piecewise continuous on $[0, \infty)$ and of exponential order; then

$$\mathscr{L}\{f * g\} = \mathscr{L}\{f(t)\}\mathscr{L}\{g(t)\} = F(s)G(s).$$

* we won't be using this theorem too often.

e.g.) $\mathscr{L}\{\int_0^t \tau \sin(t-\tau)\, d\tau\} = \mathscr{L}\{t * \sin t\}$

$= \mathscr{L}\{t\}\mathscr{L}\{\sin t\} = \frac{1}{s} \cdot \frac{1}{s^2+1}$

Inverse Form of the Convolution Theorem The convolution theorem is sometimes useful in finding the inverse Laplace transform of a product of two Laplace transforms. From Theorem 7.9 we have

$$\mathscr{L}^{-1}\{F(s)G(s)\} = f * g. \qquad (6)$$

* we'll use the inverse form of the convolution theorem a lot

* product to convolution

e.g.). $\mathscr{L}^{-1}\{\frac{1}{(s-1)(s+4)}\}$ *Do not use partial fraction decomp.

$= \mathscr{L}^{-1}\{\frac{1}{s-1}\} * \mathscr{L}\{\frac{1}{s+4}\}$

$= e^t * e^{-4t}$ $\quad f(t) = e^{-4t} \quad g(t) = e^t$

$= \int_0^t e^{-4\tau} e^{t-\tau}\, d\tau$

$= \int_0^t e^{t-5\tau}\, d\tau = e^t \int_0^t e^{-5\tau}\, d\tau = e^t \cdot -\frac{1}{5} e^{-5\tau}\Big|_0^t = e^t \frac{e^{-5t}-1}{-5} = \frac{e^t - e^{-4t}}{5}$

#9. $\mathcal{L}\{\int_0^t e^{-\tau}\cos\tau\, d\tau\}$?

this is not in the form $g(\tau) f(t-\tau)$.

$$\int_0^t e^{-\tau}\cos\tau\, d\tau = \int_0^t e^{-\tau+t-t}\cos\tau\, d\tau$$

$$= e^{-t}\int_0^t e^{t-\tau}\cos\tau\, d\tau$$

$$= e^{-t}(e^t * \cos t).$$

(since we integrate with respect to τ, we can pull out e^{-t}.)

So $\mathcal{L}\{\int_0^t e^{-\tau}\cos\tau\, d\tau\} = \mathcal{L}\{e^{-t}(e^t * \cos t)\}$

$$= \mathcal{L}\{e^t * \cos t\}_{s\to s+1}$$ (1st translation)

$$= \mathcal{L}\{e^t\}_{s\to s+1} \cdot \mathcal{L}\{\cos t\}_{s\to s+1}$$

$$= \frac{1}{s-1}\Big|_{s\to s+1} \cdot \frac{s}{s^2+1}\Big|_{s\to s+1}$$

$$= \frac{1}{s+1-1} \cdot \frac{s+1}{(s+1)^2+1} = \frac{1}{s} \cdot \frac{s+1}{(s+1)^2+1}$$

#29. $\boxed{f*g = g*f}$

Let $u = t-\tau$ $\quad du = -d\tau.$ $\quad \tau = t-u$ $\quad \tau=0 \;\; u=t.$
$\tau = t \;\; u = 0.$

$$f*g = \int_0^t f(\tau)\,g(t-\tau)\,d\tau.$$

$$= -\int_t^0 f(t-u)\,g(u)\,du.$$

$$= \int_0^t f(t-u)\,g(u)\,du$$

$$= \int_0^t g(u)\,f(t-u)\,du$$

$$= g*f.$$

#30. $\boxed{f*(g+h) = f*g + f*h}$

$$f*(g+h) = \int_0^t f(\tau)\,[g(t-\tau)+h(t-\tau)]\,d\tau$$

$$= \int_0^t f(\tau)\, g(t-\tau)\,d\tau + \int_0^t f(\tau)\, h(t-\tau)\,d\tau$$

$$= f*g + f*h.$$

#8 $\mathcal{L}\left\{\int_0^t \cos\tau \, d\tau\right\}$

$= \mathcal{L}\left\{\int_0^t \cos\tau \cdot 1 \, d\tau\right\}$ $\quad f(t) = \cos t \quad g(t) = 1.$

$g(t-\tau) = 1.$

$= \mathcal{L}\{f(t)\} \cdot \mathcal{L}\{g(t)\}$

$= \mathcal{L}\{\cos t\} \cdot \mathcal{L}\{1\}$

$= \frac{1}{s} \mathcal{L}\{\cos t\}$

$= \frac{1}{s} \frac{s}{(s^2+1)} = \frac{1}{s^2+1}$

#10 $\mathcal{L}\{\int_0^t \tau \sin\tau \, d\tau\}$.

$f = t\sin t \quad g = 1.$

$g(t-\tau) = 1.$

$= \mathcal{L}\{f(t)\} \cdot \mathcal{L}\{g(t)\}$

$= \mathcal{L}\{t\sin t\} \cdot \mathcal{L}\{1\}$

$= \frac{1}{s} \mathcal{L}\{t\sin t\}$

$= \frac{1}{s}\left(-\frac{d}{ds}\mathcal{L}\{\sin t\}\right)$

$= \frac{1}{s}\left(-\frac{d}{ds}\left(\frac{1}{s^2+1}\right)\right)$

$= \frac{1}{s} \frac{2s}{(s^2+1)^2}$

$= \frac{2}{(s^2+1)^2}.$

\# 12. $\mathcal{L}\{\int_0^t \sin\tau \cos(t-\tau)\,d\tau\}$

$f(t) = \sin t$

$g(t) = \cos t.$

$= \mathcal{L}\{\sin t\}\mathcal{L}\{\cos t\}$

$= \frac{1}{s^2+1} \cdot \frac{s}{s^2+1}$

$= \frac{s}{(s^2+1)^2}$

\#14. $\mathcal{L}\{t\int_0^t \tau e^{-\tau}\,d\tau\} = -\frac{d}{ds}\mathcal{L}\{\int_0^t \tau e^{-\tau}\,d\tau\}$

$= -\frac{d}{ds}\left(\frac{1}{s}\mathcal{L}\{te^{-t}\}\right)$

$= -\frac{d}{ds}\left(\frac{1}{s}\left[-\frac{d}{ds}\mathcal{L}(e^{-t})\right]\right).$

$= -\frac{d}{ds}\left(\frac{1}{s}\left[-\frac{d}{ds}\frac{1}{s+1}\right]\right)$

$= -\frac{d}{ds}\left(\frac{1}{s}\frac{1}{(s+1)^2}\right) = -\frac{-[(s+1)^2 + s2(s+1)]}{s^2(s+1)^4} = \frac{(s+1)+2s}{s^2(s+1)^3} = \frac{3s+1}{s^2(s+1)^3}$

#16 $\mathcal{L}\{1 * e^{-2t}\} = \dfrac{1}{s(s+2)}$

#18 $\mathcal{L}\{t^2 * te^{t}\} = \dfrac{2}{s^3}\,\mathcal{L}\{te^{t}\}$

$= \dfrac{2}{s^3} \cdot -\dfrac{d}{ds}\dfrac{1}{s-1} = \dfrac{2}{s^3}\,\dfrac{1}{(s-1)^2}$

#20 $\mathcal{L}\{e^{2t} * \sin t\} = \mathcal{L}\{e^{2t}\}\,\mathcal{L}\{\sin t\}$

$= \dfrac{1}{s-2} \cdot \dfrac{1}{s^2+1}$

#22. $\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}F(s)\right\} = \cos 2t * f(t) = \int_0^t f(\tau)\cos 2(t-\tau)\,d\tau.$

#24. $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\cdot\frac{1}{s^2+1}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$

$= 1 * \sin t = \int_0^t \sin(t-\tau)\,d\tau$

$= \cos(t-\tau)\Big|_0^t$

$= 1 - \cos t.$

#26. $\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$

$= e^{-t} * e^{-t}$

$= \int_0^t e^{-\tau} e^{-(t-\tau)}\,d\tau = e^{-t}\int_0^t d\tau = te^{-t}.$

7.4 part 3

THEOREM 7.10 Transform of a Periodic Function

Let $f(t)$ be piecewise continuous on $[0, \infty)$ and of exponential order. If $f(t)$ is periodic with period $\boxed{T}$, then

$$\mathscr{L}\{f(t)\} = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t)\, dt. \tag{8}$$

$T :=$

$f(t)$ $[0, T)$

$f(t)$ is periodic with period $T = 2$.

$$f(t) = \begin{cases} t & 0 \le t < 1 \\ 0 & 1 \le t < 2 \end{cases}$$

is the same

$[0, 2)$, $[2, 4)$, $[4, 6)$ ....

$$\mathscr{L}\{f(t)\} = \frac{1}{1 - e^{-s \cdot 2}} \int_0^2 e^{-st} f(t)\, dt.$$

$f(t+T) = f(t)$.

$$= \frac{1}{1 - e^{-2s}} \left[\int_0^1 e^{-st} \cdot t\, dt + \int_1^2 e^{-st} \cdot 0\, dt \right]$$

$$= \frac{1}{1 - e^{-2s}} \left[\int_0^1 e^{-st}\, t\, dt \right]$$

$$= \frac{1}{1 - e^{-2s}} \left[\int_0^1 \frac{t}{-s}\, e^{-st}\, d(-st) \right]$$

$$= \frac{1}{1 - e^{-2s}} \left[\overset{u}{\frac{t}{-s}}\, e^{-st} \Big|_0^1 - \int_0^1 e^{-st}\, d\frac{t}{-s} \right] \quad dv.$$

$$= \frac{1}{1 - e^{-2s}} \left[\frac{e^{-s}}{-s} - \frac{1}{(-s)^2}\, e^{-st} \Big|_0^1 \right]$$

$$= \frac{1}{1 - e^{-2s}} \left[-\frac{e^{-s}}{s} - \frac{[e^{-s} - 1]}{s^2} \right]$$

$$= \frac{1}{1 - e^{-2s}} \left[-\frac{e^{-s}}{s} - \frac{e^{-s} - 1}{s^2} \right]$$

$$\boxed{\tilde{f}(t)=\begin{cases} t & 0\le t<1 \\ 0 & 1\le t<2 \end{cases}}$$

$$f(t)=\begin{cases} g(t) & 0\le t<a \\ h(t) & a\le t \end{cases}$$

$$f(t)=g(t)\,[1-U(t-a)]+h(t)\,U(t-a)$$

$a=1$.

$$\tilde{f}(t)=t\,[1-U(t-a)]$$

$$=t\,[1-U(t-1)]=\boxed{t-t\,U(t-1)}.$$

$T=2$.

$$\mathcal{L}\{f(t)\}=\frac{1}{1-e^{-2s}}\boxed{\int_0^2 e^{-st}\boxed{\tilde{f}(t)}\,dt}.$$

$$=\frac{1}{1-e^{-2s}}\boxed{\mathcal{L}\{\tilde{f}(t)\}}.$$

$$=\frac{1}{1-e^{-2s}}\mathcal{L}\{\underline{t-t\,U(t-1)}\}=\frac{1}{1-e^{-2s}}\left[\mathcal{L}\{t\}-\mathcal{L}\{t\,U(t-1)\}\right]$$

$\frac{1!}{s^2}$ 2nd.

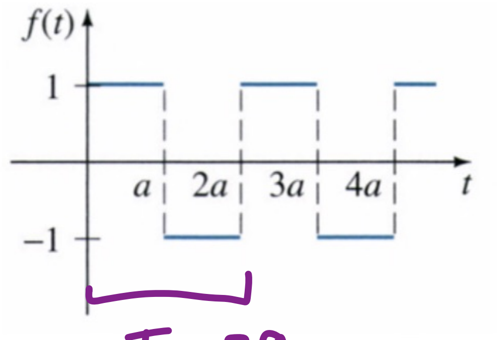
$\tilde{f}(t)=t+1$ $a=1$

$$=\frac{1}{1-e^{-2s}}\left[\frac{1}{s^2}-\mathcal{L}\{[(t-1)+1]\,U(t-1)\}\right]=\frac{1}{1-e^{-2s}}\left[\frac{1}{s^2}-e^{-s}\mathcal{L}\{t+1\}\right]$$

2nd

$$=\frac{1}{1-e^{-2s}}\left[\frac{1}{s^2}-e^{-s}\left[\frac{1}{s^2}+\frac{1}{s}\right]\right]=\frac{1-e^{-s}-se^{-s}}{s^2(1-e^{-2s})}$$

#31



T=2a

[0, 2a). [2a, 4a) ...... same.

$$\boxed{f(t) = \begin{cases} 1 & 0 \le t < a \\ -1 & a \le t < 2a \end{cases}}$$

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt.$$

$$= \frac{1}{1-e^{-s\cdot 2a}}\int_0^{2a} e^{-st} f(t)\,dt$$

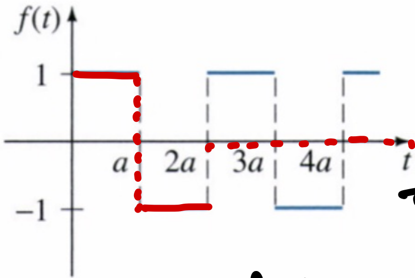
$$= \frac{1}{1-e^{-2sa}}\left[\int_0^a e^{-st}\cdot 1\,dt - \int_a^{2a} e^{-st}\,dt\right]$$

$$= \frac{1}{1-e^{-2sa}}\left[\int_0^a \frac{e^{-st}}{-s}\,d(-st) - \frac{e^{-st}}{-s}\Big|_a^{2a}\right].$$

$$= \frac{1}{1-e^{-2sa}}\left[\frac{e^{-st}}{-s}\Big|_0^a - \left[\frac{e^{-s2a}-e^{-sa}}{-s}\right]\right]$$

$$= \frac{1}{1-e^{-2sa}}\left[\frac{e^{-sa}-1}{-s} + \frac{e^{-s2a}-e^{-sa}}{s}\right] = \boxed{\frac{e^{-s2a}-2e^{-sa}+1}{s(1-e^{-2sa})}}$$

#31



$T=2a.$

$f(t+2a)=f(t).$

The unit step function can also be used to write piecewise-defined functions in a compact form. For instance, the piecewise-defined function

$$f(t)=\begin{cases} g(t), & 0\le t<a \\ h(t), & t\ge a \end{cases} \qquad (2)$$

$g(t)=1$

$h(t)=0$

can be written as

$$f(t)=g(t)-g(t)\mathcal{U}(t-a)+h(t)\mathcal{U}(t-a). \qquad (3)$$

Similarly, a function of the type

$$f(t)=\begin{cases} 0, & 0\le t<a \\ g(t), & a\le t<b \\ 0, & t\ge b \end{cases} \qquad (4)$$

$g(t)=-1$

can be written

$$f(t)=g(t)\boxed{[\mathcal{U}(t-a)-\mathcal{U}(t-b)]}. \qquad (5)$$

The window function of $f(t)$ is

$$f_{2a}(t)=\begin{cases} f(t) & 0\le t<2a \\ 0 & t\ge 2a \end{cases}$$

$$\mathcal{L}\{f(t)\}=\frac{1}{1-e^{-2as}}\mathcal{L}\{f_{2a}(t)\}.$$

$$f_{2a}(t)=\begin{cases} 1 & 0\le t<a \\ -1 & a\le t<2a \\ 0 & t\ge 2a \end{cases}$$

$$f_{2a}(t)=f_{2a_1}+f_{2a_2}$$

$$f_{2a_1}(t)=\begin{cases} 1 & 0\le t<a \\ 0 & t\ge a \end{cases} \qquad f_{2a_2}(t)=\begin{cases} 0 & 0\le t<a \\ -1 & a\le t\le 2a \\ 0 & t\ge 2a \end{cases}$$

$$f_{2a_1}=1[1-u(t-a)]+\cancel{0[u(t-a)]}$$

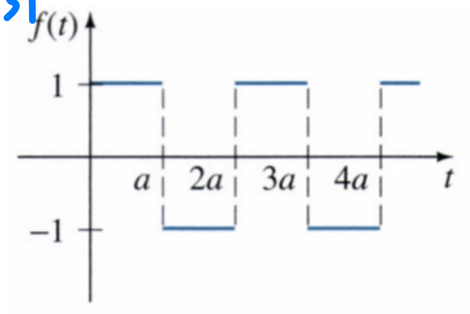
$$f_{2a_2}=(-1)[u(t-a)-u(t-2a)]$$

$$f_{2a}(t)=1-u(t-a)-[u(t-a)-u(t-2a)]$$

$$=1-2u(t-a)+u(t-2a)$$

$$\mathcal{L}\{f_{2a}(t)\}(s)$$

#31



$$\mathcal{L}\{f_{2a}(t)\}(s)$$

$$= \mathcal{L}\{1 - 2u(t-a) + u(t-2a)\}(s)$$

$$= \mathcal{L}\{1\} - 2\mathcal{L}\{u(t-a)\} + \mathcal{L}\{u(t-2a)\}$$

2nd $\mathcal{L}$ + table

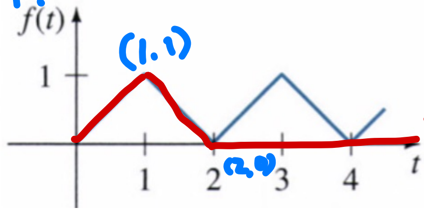
$$= \frac{1}{s} - 2e^{-as}\mathcal{L}\{1\} + e^{-2as}\mathcal{L}\{1\}$$

$$= \frac{1}{s}\left[1 - 2e^{-as} + e^{-2as}\right]$$

$$= \frac{(1-e^{-as})^2}{s}$$

$$\mathcal{L}\{f(t)\} = \frac{\mathcal{L}\{f_{2a}(t)\}}{1-e^{-2as}} = \boxed{\frac{(1-e^{-as})^2}{s(1-e^{-2as})}} = \frac{(1-e^{-as})^{\cancel{2}}}{s(1+e^{-as})\cancel{(1-e^{-as})}} = \boxed{\frac{1-e^{-as}}{s(1+e^{-as})}}$$

#34.



$T=2$. the window function of $f(t)$ is goal

$$f_2(t)=\begin{cases} t & 0\le t<1 \\ 2-t & 1\le t<2 \\ 0 & t\ge 2 \end{cases} = \begin{cases} f(t) & 0\le t\le 2 \\ 0 & t\ge 2 \end{cases}$$

triangular wave.

$$\mathcal{L}\{f(t)\}=\frac{\mathcal{L}\{f_2(t)\}}{1-e^{-2s}}$$

$$f_2(t)=\underbrace{\begin{cases} t & 0\le t\le 1 \\ 0 & t\ge 1 \end{cases}}_{f_{2a}} + \underbrace{\begin{cases} 0 & 0\le t\le 1 \\ 2-t & 1\le t<2 \\ 0 & t\ge 2. \end{cases}}_{f_{2b}}$$

$$= t[1-U(t-1)]+\cancel{0[U(t-1)]}+(2-t)[U(t-1)-U(t-2)]$$

$$= t-tU(t-1)+(2-t)U(t-1)-(2-t)U(t-2)$$

$$= t+(2-2t)U(t-1)+(t-2)U(t-2)$$

$$= t+(-2)(t-1)U(t-1)+(t-2)U(t-2).$$

$$\mathcal{L}\{f_2(t)\}=\mathcal{L}\{t\}+(-2)\mathcal{L}\{(t-1)U(t-1)\}+\mathcal{L}[(t-2)U(t-2)]$$

$a=1$ $\quad$ $a=2$

$$=\frac{1!}{s^2}-2e^{-s}\mathcal{L}\{t\}+e^{-2s}\mathcal{L}\{t\}$$

$$=\frac{1-2e^{-s}+e^{-2s}}{s^2}=\frac{(1-e^{-s})^2}{s^2}.$$

$$\mathcal{L}\{f(t)\}=\frac{1}{1-e^{-2s}}\cdot\frac{(1-e^{-s})^2}{s^2}=\frac{1}{\cancel{(1-e^{-s})}(1+e^{-s})}\cdot\frac{(1-e^{-s})^{\cancel{2}}}{s^2}=\boxed{\frac{(1-e^{-s})}{s^2(1+e^{-s})}}.$$

#34

$T=2$

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-Ts}} \int_0^T e^{-st} f(t)\,dt$$

quiz.

$$= \frac{1}{1-e^{-2s}}\left[\int_0^1 e^{-st}\cdot t\,dt + \int_1^2 e^{-st}(2-t)\,dt\right]$$

u

dv.

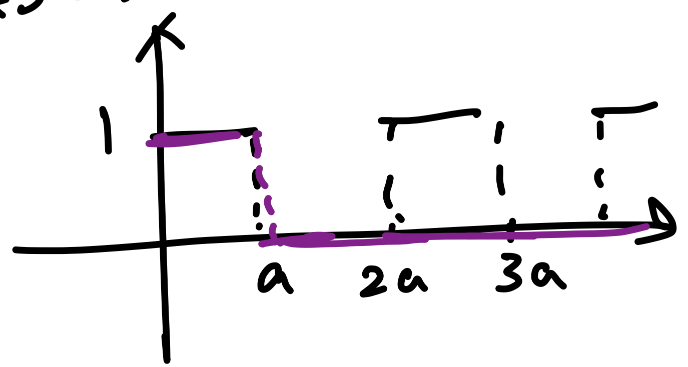
$$= \frac{1}{1-e^{-2s}}\left[\frac{e^{-st}}{-s}\cdot t\Big|_0^1 - \left(\frac{1}{-s}\right)\int_0^1 e^{-st}\,dt + \frac{e^{-st}}{-s}(2-t)\Big|_1^2 - \int_1^2 \frac{e^{-st}}{-s}\,dt\right]$$

$$= \frac{1}{1-e^{-2s}}\left[\frac{e^{-s}}{-s} - \frac{1}{(-s)^2}e^{-st}\Big|_0^1 + \left[-\frac{e^{-s}}{-s}\right] - \frac{e^{-st}}{(-s)\cdot s}\Big|_1^2\right]$$

$$= \frac{1}{1-e^{-2s}}\left[-\cancel{\frac{e^{-s}}{s}} - \frac{e^{-s}-1}{s^2} + \cancel{\frac{e^{-s}}{s}} + \frac{1}{s^2}\left[e^{-2s}-e^{-s}\right]\right]$$

$$= \frac{1}{1-e^{-2s}}\cdot\frac{1-e^{-s}+e^{-2s}-e^{-s}}{s^2} = \frac{(1-e^{-s})^2}{s^2\,\cancel{(1-e^{-s})}(1+e^{-s})} = \boxed{\frac{1-e^{-s}}{s^2(1+e^{-s})}}$$

#32.



$T = 2a.$

$$f(t) = \begin{cases} 1 & 0 \le t \le a. \\ 0. & a \le t \le 2a \end{cases}$$

$\tilde{f}(t) = 1[1 - u(t-a)] + 0[u(t-a)]$ ✓

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-s\cdot 2a}} \int_0^{2a} e^{-st} \tilde{f}(t)\,dt.$$

$$= \frac{1}{1-e^{-2as}} \mathcal{L}\{\tilde{f}(t)\}.$$

$$(1-e^{-2as}) = (1+e^{-as})(1-e^{-as})$$

$$= \frac{1}{1-e^{-2as}} \mathcal{L}[1 - u(t-a)\}.$$

$$= \frac{1}{1-e^{-2as}} [\mathcal{L}\{1\} - \mathcal{L}\{u(t-a)\}\}$$

$$= \frac{1}{1-e^{-2as}} [\frac{1}{s} - e^{-as}\mathcal{L}\{1\}] = \frac{1-e^{-as}}{(1-e^{-2as})s} = \boxed{\frac{1}{(1+e^{-as})s}}$$

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} f(t)\, dt.$$

$$= \frac{1}{1-e^{-2as}} \int_0^{a} e^{-st}\, dt + 0$$

$$= \frac{1}{1-e^{-2as}} \left[\frac{1}{-s} e^{-st} \Big|_0^a \right]$$

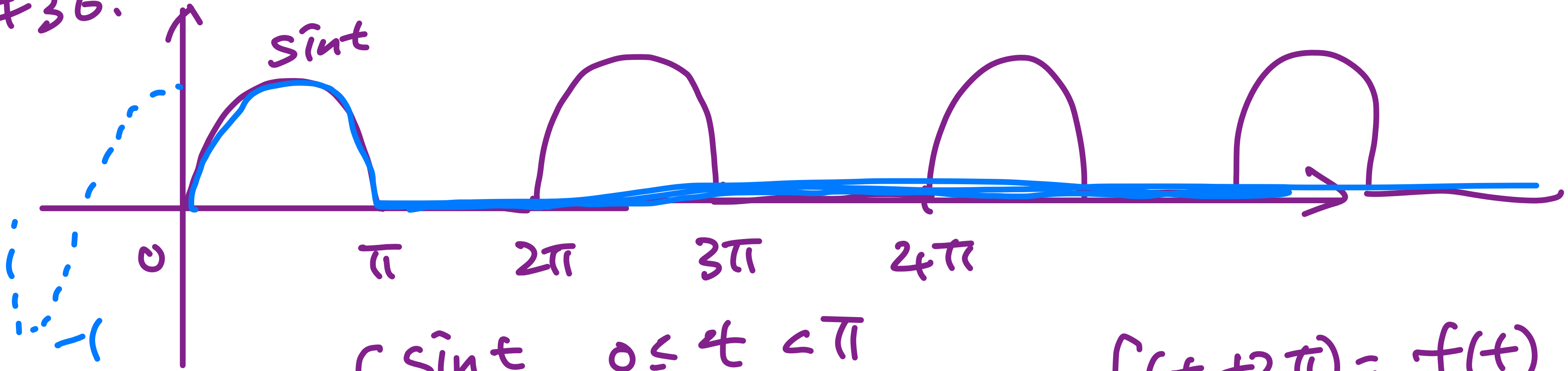
$$= \frac{1}{1-e^{-2as}} \left[\frac{1}{-s} \left[e^{-as} - 1 \right] \right]$$

$$= \frac{1-e^{-as}}{s(1-e^{-2as})}$$

$$= \frac{\cancel{1-e^{-as}}}{s\cancel{(1-e^{-as})}(1+e^{-as})} = \boxed{\frac{1}{s(1+e^{-as})}}$$

☆

#36.



$$f(t) = \begin{cases} \sin t & 0 \le t < \pi \\ 0 & \pi \le t < 2\pi \end{cases} \qquad f(t+2\pi) = f(t).$$

$$T = 2\pi$$

$$f_{2\pi}(t) = \begin{cases} f(t) & 0 \le t \le 2\pi \\ 0 & t \ge 2\pi \end{cases} = \begin{cases} \sin t & 0 \le t \le \pi \\ 0 & t \ge \pi \end{cases}$$

$$f_{2\pi}(t) = \sin t\,[1 - u(t-\pi)] + \cancel{0\,(u(t-\pi))}$$

$$= \sin t - \sin t\, u(t-\pi)$$

$$= \sin t - (-\sin(t-\pi))\, u(t-\pi)$$

$$= \sin t + \sin(t-\pi)\, u(t-\pi)$$

$$\sin(t-\pi)$$
$$= \sin t \cos(-\pi)$$
$$\cancel{+ \cos t \sin(-\pi)}$$
$$= -\sin t.$$

$$\mathcal{L}\{f_{2\pi}(t)\} = \mathcal{L}\{\sin t\} + \mathcal{L}\{\sin(t-\pi)\, u(t-\pi)\}$$

$a = \pi \quad f(t) = \sin t \quad \sin t = \sin(t-\pi)$

2nd

$$= \frac{1}{s^2+1} + e^{-\pi s}\mathcal{L}\{\sin t\} = \frac{1+e^{-\pi s}}{s^2+1}, \quad \mathcal{L}\{f(t)\} = \frac{\mathcal{L}\{f_{2\pi}(t)\}}{1-e^{-2\pi s}} = \frac{1}{(s^2+1)(1-e^{-\pi s})}$$

#36

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-2\pi s}} \boxed{\int_0^{\pi} e^{-st}\sin t\,dt} + 0$$

$$= \frac{1}{(1-e^{-2\pi s})}\frac{e^{-s\pi}+1}{s^2+1} = \frac{1}{(1-e^{-s\pi})(s^2+1)}$$

$$\boxed{\int_0^{\pi} e^{-st}\sin t\,dt} = \frac{e^{-st}}{-s}\sin t\Big|_0^{\pi} - \int_0^{\pi}\frac{e^{-st}}{-s}\cos t\,dt.$$

$$= \cancel{\frac{e^{-s\pi}}{-s}\sin\pi} - \cancel{\frac{e^{-s\cdot 0}}{-s}\sin 0} + \frac{1}{s}\int_0^{\pi} e^{-st}\cos t\,dt.$$

$$= \frac{1}{s}\left[\frac{e^{-st}}{-s}\cos t\Big|_0^{\pi} - \int_0^{\pi}\frac{e^{-st}}{-s}-\sin t\,dt\right]$$

$$= \left[\frac{1}{-s^2}e^{-st}\cos t\Big|_0^{\pi}\right] - \frac{1}{s^2}\boxed{\int_0^{\pi} e^{-st}\sin t\,dt.}$$

$$\left(1+\frac{1}{s^2}\right)\int_0^{\pi} e^{-st}\sin t\,dt = -\frac{1}{s^2}e^{-st}\cos t\Big|_0^{\pi} = -\frac{1}{s^2}\left[e^{-s\pi}(-1)-(1)\right].$$

$$\int_0^{\pi} e^{-st}\sin t\,dt = \frac{1}{1+\frac{1}{s^2}}\cdot\frac{e^{-s\pi}+1}{s^2} = \boxed{\frac{e^{-s\pi}+1}{s^2+1}}$$

7.5 applications.

THEOREM 7.8 Transform of a Derivative

If $f(t), f'(t), \ldots, f^{(n-1)}(t)$ are continuous on $[0, \infty)$ and are of exponential order and if $f^{n}(t)$ is piecewise continuous on $[0, \infty)$, then

$$\mathscr{L}\{f^{n}(t)\} = s^{n}F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0),$$

where $F(s) = \mathscr{L}\{f(t)\}$.

eg.) $\mathscr{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt = e^{-st} f(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f(t)\,dt$

$= -f(0) + s\mathscr{L}\{f(t)\}$

$= s\mathscr{L}\{f(t)\} - f(0) = sF(s) - f(0)$

eg.) $\mathscr{L}\{f''(t)\} = \int_0^\infty e^{-st} f''(t)\,dt = e^{-st} f'(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f'(t)\,dt$

$= -f'(0) + s\mathscr{L}\{f'(t)\}$

$= s^2\mathscr{L}\{f(t)\} - sf(0) - f'(0)$

e.g.) $\mathcal{L}\{3t\cos 3t + \sin 3t\}$

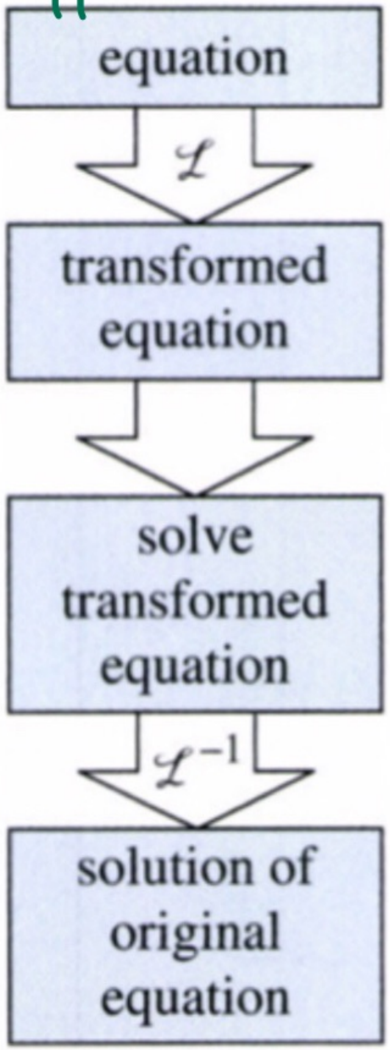
$= \mathcal{L}\{\frac{d}{dt} t\sin 3t\}$

$= s\,\mathcal{L}\{t\sin 3t\} - (0\cdot\sin 3 0)$

$= s\,\mathcal{L}\{t\sin 3t\}$

$* \frac{d}{dt}(t\sin 3t) = \sin 3t + 3t\cos 3t$

7.5 Application.



Start with a differential equation.

1. Apply laplace transform.
2. Using transform of a derivative, break down terms like $\mathcal{L}\{y''\}$.
3. End up with $\underline{\mathcal{L}\{y\}}$ = some function in terms of s.
   ↳ this will be denoted as $Y(s)$.
4. Apply inverse transform. $Y(s) = \mathcal{L}\{y\}$
   then $\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{\mathcal{L}\{y\}\} = y$.
5. End up with. $y = \mathcal{L}^{-1}$ (solve function in s).
   $=$ $\underline{\text{a function in } t}$
   ↳ this will be the solution to the given D.E. :)

For many D.E., methods we learned from the previous chapters will be easier to apply and faster when it comes to solving D.E. But this method is very general. we don't have to worry about different conditions or types of D.E.

#6. $y(0)=2$. $y'(0)=3$. Solve for $\mathcal{L}\{y(t)\}=Y(s)$. $y''+y=1$.

$$y''+y=1 \longrightarrow \mathcal{L}\{y''+y\}=\mathcal{L}\{1\}$$

...but we can also use undetermined coeffi or variation of parameters too.

$$\longrightarrow \mathcal{L}\{y''\}+\mathcal{L}\{y\}=\frac{1}{s}$$

$$\longrightarrow s^2\mathcal{L}\{y\}-sy(0)-y'(0)+\mathcal{L}\{y\}=\frac{1}{s}$$

$$\longrightarrow \mathcal{L}\{y\}(s^2+1)-2s-3=\frac{1}{s}$$

$$\longrightarrow \mathcal{L}\{y\}(s^2+1)=\frac{1}{s}+2s+3$$

$$\longrightarrow Y(s)=\frac{1+2s^2+3s}{s}\cdot\frac{1}{s^2+1}=\frac{1+2s^2+3s}{s(s^2+1)}$$

$$\mathcal{L}^{-1}(\underbrace{\mathcal{L}\{y\}}_{Y(s)})=y=\mathcal{L}^{-1}\left\{\frac{1+2s^2+3s}{s(s^2+1)}\right\} \longleftarrow \text{solution}$$

ex) $y' - 3y = e^{2t}$, $y(0) = 1$.

step1 $\mathcal{L}\{y' - 3y\} = \mathcal{L}\{e^{2t}\}$

step2 $\mathcal{L}\{y'\} - 3\mathcal{L}\{y\} = \mathcal{L}\{e^{2t}\}$

$$s\mathcal{L}\{y\} - y(0) - 3\mathcal{L}\{y\} = \frac{1}{s-2}$$

step3 $s\mathcal{L}\{y\} - 1 - 3\mathcal{L}\{y\} = \frac{1}{s-2}$.

$$sY(s) - 3Y(s) = \frac{1}{s-2} + 1 = \frac{s-1}{s-2}$$

$$Y(s)(s-3) = \frac{s-1}{s-2}$$

$$Y(s) = \frac{s-1}{(s-2)(s-3)}$$

Step4 $\mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1}\left\{\frac{s-1}{(s-2)(s-3)}\right\}$

Step5 $y(t) = \mathcal{L}^{-1}\{(s-1)/(s-2)(s-3)\}$.

$$= \mathcal{L}^{-1}\left\{\frac{-1}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{s-3}\right\}.$$ (by partial fraction decomposition)

$$= -e^{2t} + 2e^{3t}$$

$\Rightarrow$ soln of the given D.E is. $y = -e^{2t} + 2e^{3t}$

But! $y' - 3y = e^{2t}$, $y(0) = 1$.

Integrating Factor $\rightarrow$ $e^{\int -3\,dt} = e^{-3t} = \mu(t)$

$$y(t) = \frac{1}{\mu(t)} \int \mu(t) \cdot \underbrace{Q(t)}_{e^{2t}}\, dt.$$

$$= \frac{1}{e^{-3t}} \int e^{-3t} \cdot e^{2t}\, dt$$

$$= e^{3t} \int e^{-t}\, dt$$

$$= e^{3t}\left(-e^{-t} + c\right)$$

$$= -e^{2t} + ce^{3t}$$

$y(0) = -e^{2\cdot 0} + ce^{3\cdot 0} = -1 + c = 1 \Rightarrow c = 2.$

$\therefore y(t) = -e^{2t} + 2e^{3t}.$

$\longrightarrow$ since this D.E is a linear DE. using integrating factor, we can solve the linear e.q. and solve for c faster! and easier! :)

e.g.) $y' - y = 1. \quad y(0) = 0.$

$$\mathcal{L}\{y'\} - \mathcal{L}\{y\} = \mathcal{L}\{1\} = \frac{1}{s}.$$

$$s\mathcal{L}\{y\} - y(0) - \mathcal{L}\{y\} = \frac{1}{s}.$$

$$s\mathcal{L}\{y\} - \mathcal{L}\{y\} = \frac{1}{s}.$$

$$\mathcal{L}\{y\} = \frac{1}{s(s-1)} = \frac{1}{s-1} - \frac{1}{s},$$

$$y = \mathcal{L}^{-1}\{\mathcal{L}\{y\}\} = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = e^{t} - 1.$$

e.g.) $y'' + 5y' + 4y = 0$, $y(0) = 1$ $y'(0) = 0$.

$$\mathcal{L}\{y''\} + 5\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = 0.$$

$$s^2\mathcal{L}\{y\} - s\,y(0) - y'(0) + 5[s\mathcal{L}\{y\} - y(0)] + 4\mathcal{L}\{y\} = 0.$$

$$(s^2 + 5s + 4)\mathcal{L}\{y\} - s - 5 = 0.$$

$$\mathcal{L}\{y\} = \frac{s+5}{s^2+5s+4} = \frac{s+5}{(s+1)(s+4)} = \frac{\cancel{s+4}}{\cancel{(s+4)}(s+1)} + \frac{1}{(s+1)(s+4)}$$

$$= \frac{1}{s+1} + \frac{A}{s+4} + \frac{B}{s+1}$$

$1 = A(s+1) + B(s+4)$

$(A+B) = 0. \Rightarrow A = -B$

$A + 4B = 1 \Rightarrow -B + 4B = 1 \Rightarrow B = \frac{1}{3}.$

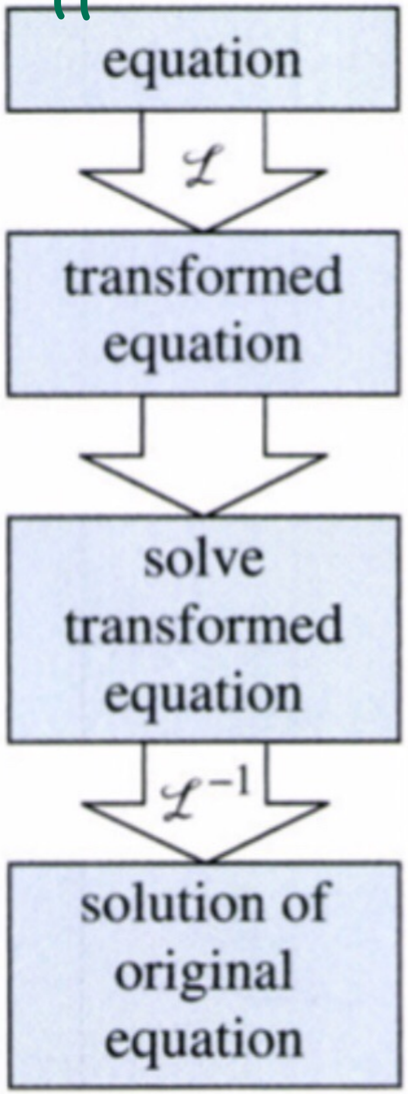
$A = -\frac{1}{3}$

$$= \frac{1}{s+1} + \frac{-\frac{1}{3}}{s+4} + \frac{\frac{1}{3}}{s+1}$$

$$= \frac{4}{3}\frac{1}{s+1} - \frac{1}{3}\frac{1}{s+4}.$$

$$y = \mathcal{L}^{-1}\mathcal{L}\{y\} = \frac{4}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\} = \frac{4}{3}e^{-t} - \frac{1}{3}e^{-4t}$$

7.5 Application.



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   ↳ this will be denoted as $Y(s)$.
4. Apply inverse transform. $Y(s) = \mathcal{L}\{y\}$
   then $\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{\mathcal{L}\{y\}\} = y$
5. End up with. $y = \mathcal{L}^{-1}$ (solve function in s).
   $=$ a function in t
   ↳ this will be the solution to the given D.E. :)

For many D.E., methods we learned from the previous chapters will be easier to apply and faster when it comes to solving D.E. But this method is very general. We don't have to worry about different conditions or types of D.E.

#6. $y(0)=2$, $y'(0)=3$. Solve for $\mathcal{L}\{y(t)\}=Y(s)$. $y''+y=1$.

$$y''+y=1 \longrightarrow \mathcal{L}\{y''+y\}=\mathcal{L}\{1\}$$

$$\longrightarrow \mathcal{L}\{y''\}+\mathcal{L}\{y\}=\frac{1}{s}$$

$$\longrightarrow s^2\mathcal{L}\{y\}-sy(0)-y'(0)+\mathcal{L}\{y\}=\frac{1}{s}$$

$$\longrightarrow \mathcal{L}\{y\}(s^2+1)-2s-3=\frac{1}{s}$$

$$\longrightarrow \mathcal{L}\{y\}(s^2+1)=\frac{1}{s}+2s+3.$$

$$\longrightarrow Y(s)=\frac{1+2s^2+3s}{s}\cdot\frac{1}{s^2+1}=\frac{1+2s^2+3s}{s(s^2+1)}$$

... but we can also use undetermined coeffi or variation of parameters too.

$$\mathcal{L}^{-1}(\underbrace{\mathcal{L}\{y\}}_{Y(s)})=y=\mathcal{L}^{-1}\left\{\frac{1+2s^2+3s}{s(s^2+1)}\right\} \longleftarrow \text{solution}$$

ex) $y' - 3y = e^{2t}$, $y(0) = 1$.

step1 $\mathcal{L}\{y' - 3y\} = \mathcal{L}\{e^{2t}\}$

step2 $\underbrace{\mathcal{L}\{y'\}} - 3\mathcal{L}\{y\} = \underbrace{\mathcal{L}\{e^{2t}\}}$

$\underbrace{s\mathcal{L}\{y\} - y(0)} - 3\mathcal{L}\{y\} = \underbrace{\frac{1}{s-2}}$

step3 $s\mathcal{L}\{y\} - 1 - 3\mathcal{L}\{y\} = \frac{1}{s-2}$.

$sY(s) - 3Y(s) = \frac{1}{s-2} + 1 = \frac{s-1}{s-2}$

$Y(s)(s-3) = \frac{s-1}{s-2}$

$Y(s) = \frac{s-1}{(s-2)(s-3)}$

Step4 $\mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1}\left\{\frac{s-1}{(s-2)(s-3)}\right\}$

Step5 $y(t) = \mathcal{L}^{-1}\{(s-1)/(s-2)(s-3)\}$

$= \mathcal{L}^{-1}\left\{\frac{-1}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{s-3}\right\}$. (by partial fraction decomposition)

$= -e^{2t} + 2e^{3t}$

$\rightarrow$ soln of the given D.E is. $y = -e^{2t} + 2e^{3t}$

But! $y' - 3y = e^{2t}$, $y(0) = 1$.

→ since this D.E is a linear D.E. using integrating factor, we can solve the linear e.q. and solve for c faster! and easier! :)

Integrating Factor → $e^{\int -3\,dt} = e^{-3t} = \mu(t)$

$$y(t) = \frac{1}{\mu(t)} \int \mu(t) \cdot \underline{Q(t)}\,dt.$$

e^{2t}

$$= \frac{1}{e^{-3t}} \int e^{-3t} \cdot e^{2t}\,dt$$

$$= e^{3t} \int e^{-t}\,dt$$

$$= e^{3t}(-e^{-t} + c)$$

$$= -e^{2t} + ce^{3t}$$

$y(0) = -e^{2\cdot 0} + ce^{3\cdot 0} = -1 + c = 1 \Rightarrow c = 2.$

$$\therefore y(t) = -e^{2t} + 2e^{3t}.$$

ex) Solve $y'' - 6y' + 9y = t^2 e^{3t}$, $y(0) = 2$, $y'(0) = 6$.

$$\mathcal{L}\{y''\} - 6\mathcal{L}\{y'\} + 9\mathcal{L}\{y\} = \mathcal{L}\{t^2 e^{3t}\}$$

$$\mathcal{L}\{y''\} = s^2 Y(s) - s\,y(0) - y'(0) = s^2 Y(s) - 2s - 6$$

$$\mathcal{L}\{y'\} = sY(s) - y(0) = sY(s) - 2.$$

$$\mathcal{L}\{y\} = Y(s)$$

$$\mathcal{L}\{t^2 e^{3t}\} = \mathcal{L}\{t^2\}_{s \to s-3} = \frac{2!}{s^{2+1}}\Big|_{s \to s-3} = \frac{2!}{(s-3)^3}$$ (1st translation thm).

$$\Rightarrow (s^2 Y(s) - 2s - 6) - 6(sY(s) - 2) + 9(Y(s)) = \frac{2!}{(s-3)^3}$$

$$Y(s)(s^2 - 6s + 9) - 2s - 6 + 12 = \frac{2!}{(s-3)^3}$$

$$Y(s)(s^2 - 6s + 9) = \frac{2}{(s-3)^3} + 2s - 6$$

$$Y(s) = \left(\frac{2}{(s-3)^3} + 2s - 6\right)\frac{1}{s^2 - 6s + 9} = \left(\frac{2}{(s-3)^3} + 2(s-3)\right)\left(\frac{1}{(s-3)^2}\right)$$

$$= \frac{2}{(s-3)^5} + \frac{2(s-3)}{(s-3)^2} = \frac{2}{(s-3)^5} + \frac{2}{(s-3)}$$

to be continue --->

continue from previous page

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{2}{(s-3)^5}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{s-3}\right\}$$

$$y(t) = 2\mathcal{L}^{-1}\left\{\frac{1}{s^5}\Big|_{s\to s-3}\right\} + 2e^{3t}$$ apply 1st translation.

$$= 2\cdot\frac{1}{4!}\mathcal{L}^{-1}\left\{\frac{4!}{s^5}\Big|_{s\to s-3}\right\} + 2e^{3t}$$

$$= \frac{2}{4!}e^{3t}t^4 + 2e^{3t}$$

THEOREM 7.3 Some Inverse Transforms

(a) $1 = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}$

(b) $t^n = \mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\}, n = 1, 2, 3, \ldots$ (c) $e^{at} = \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\}$

(d) $\sin kt = \mathcal{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\}$ (e) $\cos kt = \mathcal{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\}$

(f) $\sinh kt = \mathcal{L}^{-1}\left\{\frac{k}{s^2-k^2}\right\}$ (g) $\cosh kt = \mathcal{L}^{-1}\left\{\frac{s}{s^2-k^2}\right\}$

Inverse Form of the First Translation Theorem The inverse form of Theorem 7.5 can be written

$$\mathcal{L}^{-1}\{F(s-a)\} = \mathcal{L}^{-1}\{F(s)|_{s\to s-a}\} = e^{at}f(t), \tag{1}$$

where $f(t) = \mathcal{L}^{-1}\{F(s)\}$.

* we can also use undetermined coeff (4.4) or variation of parameter (4.7) as well.

ex) $y' + y = f(t)$, $f(t) = \begin{cases} 0 & 0 \le t < 1 \\ 5 & t \ge 1 \end{cases}$, $y(0) = 0$.

$$f(t) = 0 - 0\,u(t-1) + 5\,u(t-1) = 5\,u(t-1)$$

$$\mathcal{L}\{y'\} + \mathcal{L}\{y\} = \mathcal{L}\{5\,u(t-1)\}.$$

$$(2\mathcal{L}\{y\} - y(0)) + \mathcal{L}\{y\} = 5\,\mathcal{L}\{u(t-1)\}$$

$$sY(s) - 0 + Y(s) = 5\left(\frac{e^{-s}}{s}\right)$$

$$Y(s)(s+1) = \frac{5e^{-s}}{s}$$

$$Y(s) = \frac{5e^{-s}}{s(s+1)}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{5e^{-s}}{s(s+1)}\right\}$$

$$y(t) = 5\,\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s(s+1)}\right\}$$

$$= 5\,\mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\}_{t \to t-1} u(t-1).$$

$$= 5\,\mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s+1}\right\}_{t \to t-1} u(t-1)$$

$$= 5(1 - e^{-t})_{t \to t-1}\, u(t-1) = 5(1 - e^{-(t-1)})\,u(t-1) = 5(1 - e^{1-t})\,u(t-1)$$

* linear eq.

we can use method for linear eq. check page 59. example 7.

Solution is also in piece wise function as well.

23) $y'' + 4y = \sin t \; \mathcal{U}(t-2\pi)$, $y(0) = 1$, $y'(0) = 0$.

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = \mathcal{L}\{\sin(t)\,\mathcal{U}(t-2\pi)\}$$

$$s^2\mathcal{L}\{y\} - s\,y(0) - y'(0) + 4\mathcal{L}\{y\} = \mathcal{L}\{\underline{\sin(t-2\pi)}\;\mathcal{U}(t-2\pi)\}$$

$\sin(t \pm 2n\pi) = \sin(t)$
for all integer n.

$$s^2\mathcal{L}\{y\} - s + 4\mathcal{L}\{y\} = e^{-2\pi s}\mathcal{L}\{\sin t\}$$

$$\mathcal{L}\{y\}(s^2+4) = e^{-2\pi s}\left(\frac{1}{s^2+1}\right) + s$$

$$\mathcal{L}\{y\} = \left[e^{-2\pi s}/s^2+1 + s\right]\frac{1}{s^2+4} = \frac{e^{-2\pi s}}{(s^2+1)(s^2+4)} + \frac{s}{s^2+4}$$

$$\mathcal{L}^{-1}\{\mathcal{L}\{y\}\} = \mathcal{L}^{-1}\left\{e^{-2\pi s}/(s^2+1)(s^2+4)\right\} + \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}$$

$$y = \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+4)}\right\}_{t\to t-2\pi}\mathcal{U}(t-2\pi) + \cos 2t.$$

$$= \mathcal{L}^{-1}\left\{\frac{B}{s^2+1} + \frac{A}{s^2+4}\right\}_{t\to t-2\pi}\mathcal{U}(t-2\pi) + \cos 2t$$

$$= \left[\mathcal{L}^{-1}\left\{\frac{1}{3}\frac{1}{s^2+1}\right\}_{t\to t-2\pi} + \mathcal{L}^{-1}\left\{-\frac{1}{3}\frac{1}{s^2+4}\right\}_{t\to t-2\pi}\right]\mathcal{U}(t-2\pi) + \cos 2t$$

$$= \left[\frac{1}{3}\sin t_{t\to t-2\pi} - \frac{1}{3}\,\frac{1}{2}\sin 2t_{t\to t-2\pi}\right]\mathcal{U}(t-2\pi) + \cos 2t$$

$$= \left[\frac{1}{3}\sin(t-2\pi) - \frac{1}{6}\sin 2(t-2\pi)\right]\mathcal{U}(t-2\pi) + \cos 2t$$

For $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+4)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}$

$$= \sin t * \frac{1}{2}\sin 2t$$

$$= \int_0^t \sin(t-\tau)\,\frac{1}{2}\sin(2\tau)\,d\tau.$$

$$= \int_0^t (\sin t\cos\tau - \cos t\sin\tau)\,\frac{1}{2}\,2\sin\tau\cos\tau\,d\tau$$

$$= \int_0^t \sin t\cos^2\tau\sin\tau - \cos t\sin^2\tau\cos\tau\,d\tau$$

① $u = \cos\tau$

② $u = \sin\tau$

$$= \sin t\underbrace{\int -u^2\,du}_{①} - \cos t\underbrace{\int u^2\,du}_{②}$$

$$= -\frac{1}{3}\sin t\cos^3\tau - \frac{1}{3}\cos t\sin^3\tau \Big|_0^t$$

$$= -\frac{1}{3}\sin t\cos^3 t - \frac{1}{3}\cos t\sin^3 t - \left(-\frac{1}{3}\sin t\right)$$

$$= -\frac{1}{3}\,\frac{1}{2}\sin 2t\cos^2 t - \frac{1}{3}\,\frac{1}{2}\sin 2t\sin^2 t + \frac{1}{3}\sin t$$

$$= -\frac{1}{6}\sin 2t(\cos^2 t + \sin^2 t) + \frac{1}{3}\sin t$$

$$= -\frac{1}{6}\sin 2t + \frac{1}{3}\sin t$$

(8). $y'''' - y = t \qquad y(0)=0,\ y'(0)=0,\ y''(0)=0,\ y'''(0)=0$

$$\mathcal{L}\{y''''\} - \mathcal{L}\{y\} = \mathcal{L}\{t\}$$

$$s^4\mathcal{L}\{y\} - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) - \mathcal{L}\{y\} = \frac{1}{s^2}$$

$$\mathcal{L}\{y\}(s^4-1) = \frac{1}{s^2}$$

$$\mathcal{L}\{y\} = \frac{1}{s^2}\,\frac{1}{s^4-1}$$

$$\mathcal{L}\{y\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2(s^4-1)}\right\}$$

$$\mathcal{L}^{-1}\{y\} = \mathcal{L}^{-1}\left\{\frac{1}{2(s^2+1)} - \frac{1}{s^2} - \frac{1}{4(s+1)} + \frac{1}{4(s-1)}\right\}$$

$$y = \frac{1}{2}\sin t - t - \frac{1}{4}e^{-t} + \frac{1}{4}e^{t}$$

$$\mathcal{L}\{y\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s^4-1}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} * \left(\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s^2-1}\right\}\right)$$

$$= t * (\underline{\sin t * \sinh t})$$

this tells you not to use convolution theorem.